CROSS-EFFECTS OF HOMOTOPY FUNCTORS AND SPACES OF TREES

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WARNING: This paper is a draft and has not been fully checked for errors.

1. Total homotopy fibres of cubes of based spaces

Definition 1.1 (Cubes)**.** Given a finite set S, denote by $\mathcal{P}(S)$ the poset of subsets of S and by $\mathcal{P}_0(S)$ the poset of *proper* subsets of S. An S*-cube* of based spaces is a functor $\mathcal{P}(S) \to \mathcal{T}$. We say that an S-cube $\mathcal{X}$ is *reduced* if $\mathcal{X}(S) = *$. Notice that reduced cubes correspond to functors $\mathcal{P}_0(S) \to \mathcal{T}$.

In [2, Definition 1.1], Goodwillie defines the 'total homotopy fibre' of a cube of based spaces. He provides several equivalent descriptions. We will interpret one of these descriptions as an end computed over $\mathcal{P}(S)$.

Definition 1.2 (Total homotopy fibre (Goodwillie))**.** Firstly, given a finite set T denote by I^T the space of functions from $f : T \to I$, where T has the discrete topology and I is the unit interval. This space is a topological cube, the product of T copies of I. Following [2] we write $(I^T)_1$ for the subspace of I^T consisting of functions f such that $f(t) = 1$ for some $t \in T$. Write $\underline{I^T}$ for the quotient $I^T/(I^T)_1$, considered as a based space. (Notice that $\underline{I^\emptyset} = S^0$ because $I^\emptyset$ is the one-point space and $(I^\emptyset)_1$ is the empty subspace.)

If we now restrict to subsets of a fixed S these quotients give us a functor $\underline{I^\bullet} : \mathcal{P}(S) \to \mathcal{T}$ (that is, an S-cube). If $U \subset T \subset S$ then we have an inclusion

$$I^U \to I^T$$

that extends a function on U to T by setting it to be zero on the elements of $T - U$. This maps $(I^U)_1$ into I^T_1 and so yields a map $\underline{I^U} \to \underline{I^T}$.

Now let $\mathcal{X}$ be another S-cube, that is a functor $\mathcal{P}(S) \to \mathcal{T}$. The definition of the total homotopy fibre given by Goodwillie is then precisely the end

$$\operatorname{thofib} \mathcal{X} := \int_{T \in \mathcal{P}(S)} \mathrm{Map}_*(\underline{I^T}, \mathcal{X}(T)).$$

Here, Map_* denotes the space of basepoint-preserving maps. Thus a point in the total homotopy fibre consists of maps $I^T \to \mathcal{X}(T)$ such that $(I^T)_1$ maps to the basepoint and satisfying compatibility conditions.

Proposition 1.3. *Let $\mathcal{X}$ be a reduced cube of based spaces. Then the total homotopy fibre of $\mathcal{X}$ is given by an end computed over $\mathcal{P}_0(S)$:*

$$\operatorname{thofib} \mathcal{X} := \int_{T \in \mathcal{P}_0(S)} \mathrm{Map}_*(\underline{I^T}, \mathcal{X}(T)).$$

Proof. It is easy to check that the ends over $\mathcal{P}(S)$ and $\mathcal{P}_0(S)$ are isomorphic when $\mathcal{X}(S) = *$. □

Our goal is to introduce a new definition of the total homotopy fibre of a reduced cube of based spaces. This will be homotopy equivalent to that of Goodwillie. It is constructed in a similar way, as an end computed over $\mathcal{P}_0(S)$. However, we replace the spaces $\underline{I^T}$ with spaces of weighted labelled trees.

Definition 1.4 (Trees)**.** We use the same species of trees as in [1]. These are finite contractible directed graphs in which no two edges have the same initial vertex and for which there is a unique final edge, for example: The initial vertices of the tree will be called the

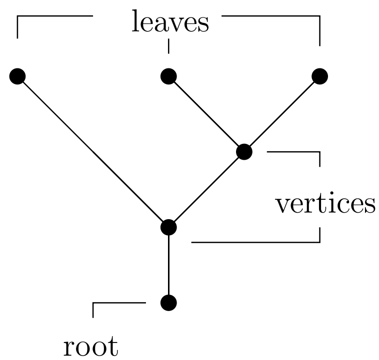


FIGURE 1. Terminology for trees

leaves and the unique final vertex of the tree will be the *root*. From now on, *vertex* will mean one of the other (internal) vertices of the tree. We also insist that our vertices have at least two input edges (and hence valence at least 3). A tree is *binary* if all its vertices have exactly two input edges. The *root edge* is the edge whose final vertex is the root and the *leaf edges* are those whose initial vertices are the leaves. The other edges are *internal edges*.

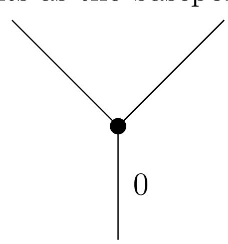
Let S be a finite set. The tree τ is said to be *labelled by a partition of* S if we are given a partition of S into nonempty subsets and a bijection between the set of those subsets and the leaves of τ.

A *weighting* on a tree τ is an assignment of non-negative 'lengths' to the edges of τ in such a way that the total 'distance' from the root to each leaf is equal to 1.

We will now use trees to define a functor $w_{S-\bullet} : \mathcal{P}_0(S) \to \mathcal{T}$ that will play the rôle of the functor $\underline{I^\bullet}$ in the definition of the total homotopy fibre.

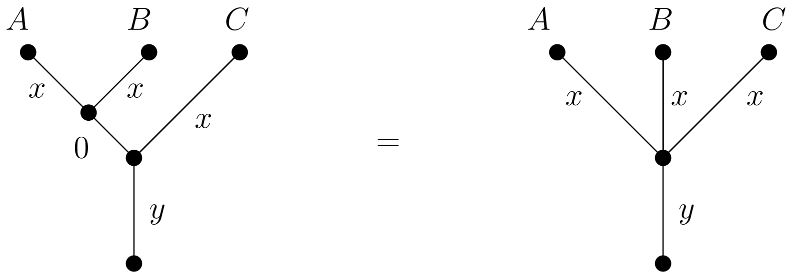
Definition 1.5. Fix a finite set S and let T be a nonempty subset of S. Then let w_T (or $w_{T \subset S}$ if S needs to be made explicit) be the space of weighted trees labelled by partitions of S that have T *contained in* one of the pieces of the partition. Trees of different shapes are identified with each other via weightings that have edges of length zero. The following diagrams show the identifications that we make:

(1) root edge of length zero counts as the basepoint:

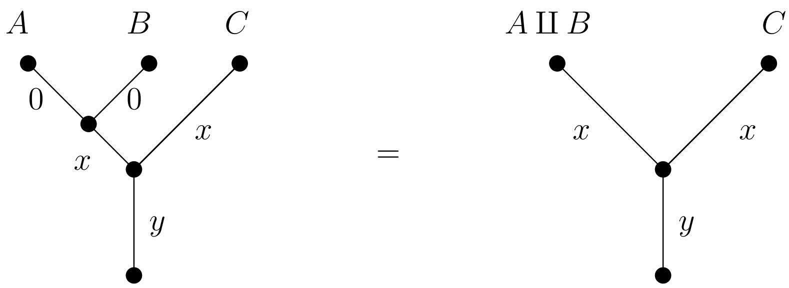


$$= \quad *$$

(2) internal edges of length zero can be collapsed:



(3) leaf edges of length zero are collapsed with labellings for the leaves joined together:



The set of weightings on a fixed tree has a topology given as a subset of the space of functions from the edges of the tree to $[0, 1]$. The topology on w_T is determined by these spaces of weightings subject to the above identifications.

Remark 1.6. If $T = S$, the partition we are labelling our trees with must be trivial. In this case, there is only one possible tree (the tree with one edge) and only one weighting. The basepoint in this case is disjoint (we really have quotiented by the empty set) and so $w_S = S^0$.

When $U \subset T \subset S$ there is a natural inclusion $w_{S-U} \to w_{S-T}$. Since $S - T \subset S - U$, a weighted tree labelled by a partition that has $S - U$ contained in one of its pieces is also labelled by a partition that has $S - T$ contained in one of its pieces. We therefore obtain a functor $w_{S-\bullet} : \mathcal{P}_0(S) \to \mathcal{T}$ as promised.

Remark 1.7. Allowing T to be empty in Definition 1.5 we could define this on the whole of $\mathcal{P}(S)$. However, constructions we make later will not be possible for T empty and so we restrict now to $\mathcal{P}_0(S)$.

Definition 1.8 (Tree-based total homotopy fibre of a reduced cube)**.** Let $\mathcal{X}$ be a reduced cube of based spaces. The *tree-based total homotopy fibre* of $\mathcal{X}$ is the end

$$\operatorname*{thofib}_{tree} \mathcal{X} := \int_{T \in \mathcal{P}_0(S)} \mathrm{Map}_*(w_{S-T}, \mathcal{X}(T)).$$

Justification for calling this the total homotopy fibre at all will follow.

Our next task is to construct maps relating the standard and tree-based total homotopy fibres. To do this we relate the spaces $\underline{I^T}$ and w_{S-T}. The following proposition is the main substance of this paper.

Proposition 1.9. *Fix a finite set S. There are natural transformations*

$$\iota : \underline{I^\bullet} \to w_{S-\bullet}$$

and

$$\pi : w_{S-\bullet} \to \underline{I^{\bullet}}$$

of functors $\mathcal{P}_0(S) \to \mathcal{T}$ *such that*

$$\pi\iota : \underline{I^T} \to \underline{I^T}$$

is the identity and

$$\iota\pi : w_{S-T} \to w_{S-T}$$

is homotopic to the identity for $T \in \mathcal{P}_0(S)$*. Moreover this homotopy is natural in* $T \in \mathcal{P}_0(S)$*.*

Proof. We first construct π. Given $T \in \mathcal{P}_0(S)$, a point in w_{S-T} is a weighted tree τ labelled by a partition λ of S such that one of the pieces contains the nonempty subset $S - T$. This tree determines a function $T \to I$ as follows. Given $t \in T$ we look at the distance from the leaf at which the branch labelled by t (or rather by the piece of the partition containing t) meets the branch labelled by $S - T$ (if t is in the same piece of the partition as $S - T$ then the distance is zero). For short, we call this *the distance at which* t *meets* $S - T$. This distance is an element of I. Doing this for each $t \in T$ determines a point in I^T and hence a point $\pi(\tau)$ in $\underline{I^T}$. We will refer t

We must first check that this construction is well-defined under the identifications of trees made in the definition of w_{S-T}. First suppose the root edge of τ has length zero. Then the distance from the vertex adjoining the root edge to a leaf will be 1. This vertex must have at least two incoming branches, one of which leads to the leaf labelled by $S - T$. Pick a t that labels a leaf that is at the end of one of the other incoming branches. Then the distance at which t meets $S - T$ is 1 and hence $\pi(\tau)$ is the basepoint in $\underline{I^T}$. This shows that the basepoint in w_{S-T} maps to the basepoint in $\underline{I^T}$.

Next suppose one of the internal edges of τ has length zero. The distances at which the t-branches meet the $(S-T)$-branch will not be affected by collapsing that internal edge. So the map π is well-defined under the second identification of Definition 1.5. Finally, suppose some of the leaf edges in τ have length zero. Then the t in pieces of the partition labelling those leaves will meet $S - T$ at distance 0. This is also the case if we collapse all these leaves to one labelled by the union of the original labels. This completes the check that we have a well-defined base-point preserving map $w_{S-T} \to I^T$.

To see that these maps form a natural transformation take an inclusion $T \subset T'$. We must check that the diagram

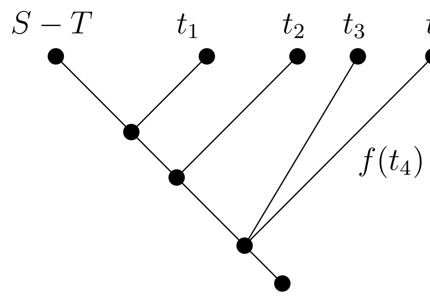
$$\begin{array}{ccc} w_{S-T} & \xrightarrow{\pi} & \underline{I^T} \\ \downarrow & & \downarrow \\ w_{S-T'} & \xrightarrow{\pi} & \underline{I^{T'}} \end{array}$$

commutes. To see this, take a point in w_{S-T}. The image in $w_{S-T'}$ is the same tree. It therefore has the same distances for points in T and distances equal to zero for points in $T' - T$. But the map $I^T \to I^{T'}$ is extension by zero so the diagram does commute.

We therefore have constructed the natural transformation

$$\pi : w_{S-\bullet} \to \underline{I^{\bullet}}.$$

For ι we must take a point in $\underline{I^T}$ and construct a suitable tree to represent a point in w_{S-T}. Suppose our point in $\underline{I^T}$ is not the basepoint - it is therefore represented by a unique function $f : T \to I$ with $f(t) \neq 1$ for all $t \in T$. Construct a tree as follows. The tree will have one leaf labelled by $S - T$ and a leaf labelled by $\{t\}$ for each $t \in T$. We take the 'trunk' of the tree to be the path from the $S - T$ leaf to the root and adjoin a branch for each $t \in T$ at a distance $f(t)$ from the leaf. For example:



The basepoint in $\underline{I^T}$ is given by functions that take the value 1 on some t. These would determine trees in which the root edge had length zero which would be the basepoint in w_{S-T}. This process therefore gives us a continuous basepoint-preserving map $\underline{I^T} \to w_{S-T}$.

To see that these maps form a natural transformation we check that the diagram

$$\begin{tikzcd}
\underline{I^T} \arrow[r, "\iota"] \arrow[d] & w_{S-T} \arrow[d] \\
\underline{I^{T'}} \arrow[r, "\iota"] & w_{S-T'}
\end{tikzcd}$$

commutes for $T \subset T'$. To see this notice that if a point in $\underline{I^{T'}}$ maps t to zero for all $t \in T' - T$, then the corresponding tree has leaf edges of length zero . It is therefore equal (in $w_{S-T'}$) to a tree with a leaf labelled with $S - T$ and branches for $t \in T$ attached. Hence the required diagram commutes.

Having constructed the maps in question we turn to the composites. It is clear that $\pi\iota$ is equal to the identity on $\underline{I^T}$. The composite $\iota\pi$ is more complicated. However we will define a homotopy from the identity on w_{S-T} to $\iota\pi$. To do this, first notice that any point in w_{S-T} has a representative tree with leaves labelled by $S - T$ and the one-point sets $\{t\}$ for $t \in T$. We obtain this representative by using identification 3 of Definition 1.5 to break up the leaves as required. Now notice that the composite $\iota\pi(\tau)$ is precisely the tree obtained from this representative of $\tau \in w_{S-T}$ by collapsing the internal edges that are not part of the $S - T$ branch to zero length, increasing the lengths of the leaf edges as necessary (the $S - T$ branch and the vertices on it remain fixed throughout this process). This collapsing can be done continuously and determines our homotopy from the identity to $\iota\pi$. $\square$

Remark 1.10. In the constructions of the maps ι and π we use extensively the fact that T is a *proper* subset of S. These constructions do not work on the whole of $\mathcal{P}(S)$.

Corollary 1.11. *The tree-based total homotopy fibre of a reduced cube $\mathcal{X}$ of based spaces is homotopy equivalent to the standard total homotopy fibre.*

Proof. The natural transformations ι and π induce maps

$$\iota^* : \underset{tree}{\operatorname{thofib}} \mathcal{X} \to \operatorname{thofib} \mathcal{X}$$

and

$$\pi^* : \operatorname{thofib} \mathcal{X} \to \operatorname*{thofib}_{tree} \mathcal{X}$$

such that $\iota^*\pi^*$ is the identity on thofib. The homotopies of Proposition 1.9 are natural in T and so induce a homotopy from $\pi^*\iota^*$ to the identity on the tree-based total homotopy fibre. Here we use that natural transformations from $w_{S_\bullet}$ to itself *continuously* induce natural transformations on the end that is the tree-based total homotopy fibre. □

2. Cross-effects of homotopy functors

In [3], Goodwillie defines the n^{th} cross-effect of a homotopy functor as the total homotopy fibre of an n-cube associated to the functor. Using the constructions of section 1 we can talk about the 'tree-based' cross-effect for the corresponding tree-based total homotopy fibre. The advantage of this is that it helps us define certain maps relating the cross-effects of a composite of two homotopy functors to the composites of the cross-effects of the individual functors. These maps are hoped to form the basis of a chain rule for the homotopy calculus.

Definition 2.1 (Cross-effects)**.** Let $F : \mathcal{T} \to \mathcal{T}$ be a functor that preserves weak equivalences of based spaces. Denote by $X \mapsto \tilde{X}$ a cofibrant relacement functor in $\mathcal{T}$. In other words, for any X, the basepoint in $\tilde{X}$ is nondegenerate.

For a finite set S, the S^{th} cross-effect of F is the following functor $F : \mathcal{T}^S \to \mathcal{T}$. Given a collection of based spaces $\{X_s\}_{s\in S}$ indexed by S, we construct an S-cube $\mathcal{X}$ of based spaces by

$$\mathcal{X}(T) := F\left(\bigvee_{s \notin T} \tilde{X}_s\right).$$

If $T \subset T'$ then the map $\mathcal{X}(T) \to \mathcal{X}(T')$ is given by collapsing the unwanted factors (those for $t \in T' - T$) of the wedge product to the basepoint and the identity on the other factors. The value of the cross-effect at the spaces $\{X_s\}$ is then the total homotopy fibre of this cube:

$$\operatorname{cr}_S F(\{X_s\}) := \operatorname{thofib} \mathcal{X}.$$

Remark 2.2. The symmetric group on S acts on the S^{th} cross-effect by permuting the input spaces. The n^{th} cross-effect (written cr_n) is the S^{th} cross-effect with $S = \{1, \dots, n\}$.

Proposition 2.3 (Goodwillie)**.** *The n^{th} derivative of a homotopy functor F is the coefficient spectrum of the symmetric multilinear functor obtained by multilinearizing the n^{th} cross-effect.*

Definition 2.4 (Tree-based cross-effects)**.** Let F be a homotopy functor such that $F(*) = *$. Then the S-cube $\mathcal{X}$ of Definition 2.1 is reduced. Then the S^{th} *tree-based cross-effect* of F is the functor $\mathcal{T}^S \to \mathcal{T}$ given by

$$\operatorname{cr}_S^{tree} F(\{X_s\}) := \operatorname*{thofib}_{tree} \mathcal{X}$$

where $\mathcal{X}$ is the S-cube of Definition 2.1.

Remark 2.5. This construction only works for F such that $F(*) = *$.

Remark 2.6. We can think of a point in the tree-based cross-effect of F as follows. For any weighted tree τ labelled by a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of S, it gives us a point in

$$F\left(\bigvee_{s \in \lambda_1} \tilde{X}_s\right) \times \cdots \times F\left(\bigvee_{s \in \lambda_r} \tilde{X}_s\right).$$

These associations must be continuous and must respect identifications of trees involving edges of length zero.

Proposition 2.7 (Goodwillie)**.** *The n^{th} derivative of a homotopy functor $F : \mathcal{T} \to \mathcal{T}$ is equivalent to the coefficient spectrum of the symmetric multilinear functor obtained by multilinearizing the n^{th} cross-effect of F.*

Corollary 2.8. *The n^{th} derivative of a homotopy functor $F : \mathcal{T} \to \mathcal{T}$ that satisfies $F(*) = *$ is equivalent to the coefficient spectrum of the multilinear functor obtained by multilinearizing the n^{th} tree-based cross-effect of F.*

Proof. This follows from the Proposition because the tree-based cross-effect is homotopy equivalent to Goodwillie's cross-effect by Corollary 1.11. □

Comparing the definition of the tree-based cross-effect with the methods used in [1] to show that the derivatives $\partial_* I$ of the identity functor I on based spaces form an operad, it seems likely that the tree-based cross-effects are appropriate for showing that the derivatives of F form a (left) module over that operad.

References

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