DERIVED KOSZUL DUALITY AND TQ-HOMOLOGY COMPLETION OF STRUCTURED RING SPECTRA

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Abstract. Working in the context of symmetric spectra, we consider algebraic structures that can be described as algebras over an operad $\mathcal{O}$. Topological Quillen homology, or TQ-homology, is the spectral algebra analog of Quillen homology and stabilization. Working in a framework for homotopical descent, the associated TQ-homology completion map appears as the unit map of an adjunction comparing the homotopy categories of $\mathcal{O}$-algebra spectra with coalgebra spectra over the associated comonad via TQ-homology. We prove that this adjunction between homotopy categories can be turned into an equivalence by replacing $\mathcal{O}$-algebras with the full subcategory of 0-connected $\mathcal{O}$-algebras. We also construct the spectral algebra analog of the unstable Adams spectral sequence that starts from the TQ-homology groups $\mathsf{TQ}_*(X)$ of an $\mathcal{O}$-algebra X, and prove that it converges strongly to $\pi_*(X)$ when X is 0-connected.

1. Introduction

The aim of this paper is to prove analogs for structured ring spectra of several foundational results for spaces concerning Bousfield-Kan completion and the unstable Adams spectral sequence. These results can also be interpreted as a version of derived Koszul duality phenomena and are related to Goodwillie calculus of the identity functor for categories of (various flavors of) ring spectra.

Ring spectra have played a crucial role in the development of algebraic topology in the last several decades. The Brown Representability Theorem shows that spectra are the representing objects for cohomology theories, but the introduction of E_n-algebras illuminated the rich internal structure many of these theories possess, clarifying, for example, the origin of power operations. In modern language, structured ring spectra are described by operads in one of the symmetric monoidal models for the stable homotopy category. Thus there are operads E_n, for $1 \leq n \leq \infty$, and an E_n-ring spectrum is a spectrum equipped with an action of E_n.

In this paper we work with algebras over an operad $\mathcal{O}$ in $\mathcal{R}$-modules, where $\mathcal{R}$ is a commutative ring spectrum; i.e., a commutative monoid object in the category $(\mathsf{Sp}^\Sigma, \otimes_S, S)$ of symmetric spectra [63, 104]. We work mostly in the category of $\mathcal{R}$-modules which we denote by $\mathsf{Mod}_{\mathcal{R}}$. For definiteness, we choose symmetric spectra as our underlying model for spectra, but our results are not limited to that context. Our main assumptions are that $\mathcal{O}[0] = *$, together with some mild cofibrancy conditions on the operad $\mathcal{O}$. This includes the examples of (non-unital) E_n-algebra spectra, and opens up applications to other operads that arise in stable homotopy theory, for example, via Goodwillie calculus. The condition on $\mathcal{O}[0]$ means that $\mathcal{O}$-algebras are non-unital. One could equivalently work with augmented algebras over an operad with a nontrivial 0-arity piece equal to $\mathcal{R}$. For a useful introduction to algebras over operads, see May [80] and Rezk [98], followed by Berger-Moerdijk

[14], Fresse [41, 42], Kapranov-Manin [68], Kelly [69], Kriz-May [70], and Markl-Shnider-Stasheff [79], for a useful starting point.

The main construction we study is topological Quillen homology, or TQ-homology. This is the precise topological analog of Quillen homology [94] in the higher algebra setting of structured ring spectra. It was introduced first by Basterra in [7], as topological André-Quillen homology (TAQ), as the derived indecomposables of the augmentation ideal of an augmented commutative ring spectrum. Further significance of this construction was made clear by Basterra-Mandell [8] who showed that the TAQ spectrum plays the role of the stabilization, or suspension spectrum. They showed that the category of "spectra" for augmented commutative R-algebras (over a commutative S-algebra R) can be identified with the category of R-modules. The "suspension spectrum" of an R-algebra is its TAQ-homology spectrum, and the "infinite loop-space" of an R-module M is the square-zero extension $R \vee M$. In fact, Basterra-Mandell worked more generally, in the context of algebras (in spectra) over an operad $\mathcal{O}$ of spaces. They constructed the stabilization of an augmented $\mathcal{O}$-algebra and showed that "spectra" can be identified with modules over a universal enveloping algebra. These results are closely related to Schwede's [102, 103] earlier study of stabilization of algebraic theories and his identification of stabilization with Quillen homology. In [78], Mandell compares the corresponding algebraic and topological contexts for Quillen homology and stabilization; the upshot is that TAQ-cohomology of a commutative ring spectrum with Eilenberg-Mac Lane coefficients can be computed as the André-Quillen cohomology of an associated E_∞ differential graded algebra. Basterra-Mandell [8], furthermore, show that TAQ-cohomology is a cohomology theory (appropriately defined via axioms analogous to the spaces context, but for structured ring spectra) and that every such cohomology theory is TAQ-cohomology with appropriate coefficients. The term "TQ-homology" appears in Basterra-Mandell [9] where they prove that for an augmented E_n-ring spectrum, the reduced version of iterated topological Hochschild homology is weakly equivalent to its associated TQ-homology, shifted by n.

Remark 1.1. André [1] and Quillen [94, 96] studied the homotopy theory of simplicial commutative algebras, in a relative setting. The homotopy meaningful Quillen homology is captured by derived abelianization, and the homology object that arises from this process is the cotangent complex, producing a relative notion of "homology" for commutative algebras now called André-Quillen homology. There is an associated André-Quillen cohomology, that arises from the cotangent complex in an analogous manner to the cohomology of spaces. These Quillen homology ideas have been exploited in Goerss-Hopkins' [49, 50] work on developing an obstruction theory for commutative spectral algebras and play a key role, in the context of augmented simplicial commutative algebras, in H.R. Miller's [86] proof of the Sullivan conjecture. The work of Goerss [48] is an extensive development, exploiting a homotopy point of view, of several properties of Quillen homology, in the same reduced setting used by Miller [86], of augmented commutative $\mathbb{F}_2$-algebras. In this context, Goerss [48] also provides an interesting comparison between Quillen homology in the reduced and relative settings, using the suspension ΣA of an augmented commutative $\mathbb{F}_2$-algebra A exploited in [86]; see also Dwyer-Spalinski [36, 11.3]. Goerss-Schemmerhorn [52] and Weibel [110] provide useful introductions to André-Quillen homology.

The stabilization interpretation of TAQ and TQ homology has been exploited in Kuhn's [71] study of the localization of André-Quillen-Goodwillie towers, which interpolates between TQ homology (or stabilization) and the identity functor. The tower is a particular instance of Goodwillie's [54] Taylor towers, which have been studied in the closely related work of Johnson-McCarthy [66], Kantorovitz-McCarthy [67], McCarthy-Minasian [83] and Minasian [87], and the subsequent work of [58] and Pereira [92]. Basterra-Mandell [8] prove that a map which induces a weak equivalence on TQ-homology is a weak equivalence if the structured ring spectra are connected; see also the closely related Whitehead theorem results in Goerss [48], Kuhn [71], Lawson [72], Livernet [73], and Schwede [103], and the Serre-type finiteness results [58] proving that certain finiteness properties for TQ-homology groups imply corresponding finiteness properties for homotopy groups.

The main construction we study is the Topological Quillen homology of an $\mathcal{O}$-algebra. As noted above, Basterra-Mandell [8] (see also Pereira [93]) show that the stabilization of the model category of $\mathcal{O}$-algebras is equivalent to the category of $\mathcal{O}[1]$-modules, with the TQ-*homology* $\mathcal{O}$-*algebra* $\mathsf{TQ}(X) \simeq U\mathsf{L}Q(X)$, and the TQ-*homology spectrum* $\mathsf{L}Q(X)$ playing the role of the suspension spectrum; here, there is a Quillen adjunction

$$\mathsf{Alg}_{\mathcal{O}} \underset{U}{\overset{Q}{\rightleftarrows}} \mathsf{Alg}_J \sim \mathsf{Mod}_{\mathcal{O}[1]}$$

in which Q is a "fattened-up" version of the indecomposables quotient (Section 3.8), Alg_J is a "fattened-up" version of the category of $\mathcal{O}[1]$-modules, U is the forgetful functor, and the indicated "$\sim$" denotes a Quillen equivalence; the "fattened-up" constructions enable the property that both U and Q preserve cofibrant objects, hence permitting iterations of UQ to be homotopy meaningful on cofibrant $\mathcal{O}$-algebras. The adjunction (Q, U) is the analog of the usual suspension spectrum and infinite loop-space adjunction $(\Sigma^\infty, \Omega^\infty)$ in the context of $\mathcal{O}$-algebras.

In this paper we use the adjunction (Q, U) to describe additional structure possessed by the TQ-homology spectrum $\mathsf{L}Q(X)$ of an $\mathcal{O}$-algebra X, and we then show the result that, under a connectivity condition, X itself can be recovered from $\mathsf{TQ}(X)$ together with this extra structure (e.g., $X \simeq X^\wedge_{\mathsf{TQ}}$). This is the spectral algebra analog of the fact (see Carlsson [22] and the subsequent work in Arone-Kankaanrinta [4]) that a simply-connected pointed space X can be recovered from the action of the derived comonad $\Sigma^\infty\Omega^\infty$ on $\Sigma^\infty X$.

Associated to the adjunction (Q, U) is the comonad $\mathsf{K} = QU\colon \mathsf{Alg}_J \to \mathsf{Alg}_J$, and recall that Q, and hence also the TQ-homology spectrum given by its derived functor $\mathsf{L}Q$, takes values in the category $\mathsf{coAlg}_{\mathsf{K}}$ of K-coalgebras. Our main result is then the following.

Theorem 1.2. *Let $\mathcal{R}$ be a (-1)-connected commutative ring spectrum, and $\mathcal{O}$ an operad in $\mathsf{Mod}_{\mathcal{R}}$ with trivial 0-ary operations and whose terms are (-1)-connected. Then the TQ-homology spectrum functor*

$$\mathsf{L}Q : \mathsf{Ho}(\mathsf{Alg}_{\mathcal{O}}) \to \mathsf{Ho}(\mathsf{coAlg}_{\mathsf{K}})$$

restricts to an equivalence between the homotopy categories of 0-connected $\mathcal{O}$-algebras and 0-connected K-coalgebras; more precisely, the natural zigzags of weak equivalences of the form (see Proposition 5.2)

$$\mathrm{Map}_{\mathsf{coAlg}_{\mathsf{K}}}(\mathsf{L}Q(X), Y) \simeq \mathrm{Map}_{\mathsf{Alg}_{\mathcal{O}}}\big(X, \mathrm{holim}_\Delta C(Y)\big) \tag{1}$$

have associated derived unit and counit maps (Definitions 5.3 and 5.5) that induce isomorphisms on the level of the homotopy categories after restriction to the full subcategories of 0*-connected cofibrant objects. Here we are assuming that* $\mathcal{O}$ *satisfies Cofibrancy Condition 2.1,* X *is cofibrant in* $\mathsf{Alg}_{\mathcal{O}}$ *and* Y *is cofibrant in the underlying category* Alg_J*; note that the* TQ*-homology* $\mathcal{O}$*-algebra* $\mathsf{TQ}(X) \simeq U\mathsf{L}Q(X)$*.*

The homotopy category of $\mathcal{O}$-algebras referred to in Theorem 1.2 is just that associated with the positive flat stable model structure on algebras over an operad, but it is well-known that categories of coalgebras rarely admit an easily understood model structure. Our homotopy theory for $\mathsf{coAlg}_{\mathsf{K}}$ is constructed explicitly, based on work of Greg Arone and the first author; we describe a topological A_∞-category whose objects are K-coalgebras and taking path components of the associated mapping spaces (Definition 4.7) then provides the morphisms in the homotopy category.

Remark 1.3. We can state our results in the context of Lurie's theory of ∞-categories in the following way. Associated to the simplicial Quillen adjunction (Q, U) is a corresponding adjunction of ∞-categories [74, 5.2.4.6]. Riehl and Verity [100] show that an adjunction of ∞-categories of the form

$$\mathsf{Q} : \mathcal{A} \rightleftarrows \mathcal{B} : \mathsf{U}$$

determines a homotopy coherent comonad K whose underlying functor is QU. If the ∞-category $\mathcal{A}$ admits suitable limits, the adjunction (Q, U) lifts to an adjunction of ∞-categories of the form

$$\bar{\mathsf{Q}} : \mathcal{A} \rightleftarrows \mathcal{C}_{\mathsf{K}} : \bar{\mathsf{C}}$$

where $\mathcal{C}_{\mathsf{K}}$ is an ∞-category of coalgebras over the comonad K and $\bar{\mathsf{C}}$ is a suitable cobar construction applied to K-coalgebras. This is dual to [100, 7.2.4]. Our Theorem 1.2 then implies that, when applied to the TQ-homology adjunction (Q, U) above, the resulting adjunction $(\bar{Q}, \bar{C})$ restricts to an equivalence between the ∞-categories of 0-connected objects on each side. In particular, what we call the *derived* unit and counit maps can be interpreted as the unit and counit of this adjunction of ∞-categories.

Remark 1.4. In Francis-Gaitsgory [40, 3.4.5] it is conjectured that replacing $\mathcal{O}$-algebras with the full subcategory of homotopy pro-nilpotent $\mathcal{O}$-algebras in (1) will induce an equivalence on the level of homotopy categories. Since in [58] it is proved that every 0-connected $\mathcal{O}$-algebra is homotopy pro-nilpotent (i.e., is the homotopy limit of a tower of nilpotent $\mathcal{O}$-algebras), our main result resolves in the affirmative the 0-connected case of [40, 3.4.5], up to a change of framework. In the notation of [40, Section 6.2.5], the comonad K is called the "Koszul dual comonad" associated to the monad $\mathcal{O}$ (i.e., the free algebra monad associated to the operad $\mathcal{O}$).

Our main result, Theorem 1.2, is also an example of derived Koszul duality phenomena. The structure of the underlying functor of the comonad K can be easily calculated from a homotopy point of view. Assume for notational simplicity that $\mathcal{O}(1) = \mathcal{R}$. Then for a 0-connected cofibrant $\mathcal{R}$-module Y, there is a zigzag of weak equivalences

$$\mathsf{K}(Y) \simeq \prod_{t \geq 1} B(\mathcal{O})[t] \wedge_{\Sigma_t} Y^{\wedge t}$$

provided that $\mathcal{R}, \mathcal{O}$ are (-1)-connected, where $B(\mathcal{O}) \simeq |\operatorname{Bar}(I, \mathcal{O}, I)|$ denotes the cooperad [23, 24] associated to the operad $\mathcal{O}$ and I denotes the initial operad (see

Section 3.1). Thus a K-coalgebra structure on Y can be viewed, up to a zigzag of weak equivalences, as a collection of suitable morphisms of the form

$$Y \rightarrow B(\mathcal{O})[t] \wedge_{\Sigma_t} Y^{\wedge t}, \qquad t \geq 1.$$

In other words, Y "looks like" a *divided power* coalgebra with t-ary co-operations parametrized by the right $\mathcal{R}[\Sigma_t]$-module $B(\mathcal{O})[t]$. This is a divided power structure because the target of these maps consists of the coinvariants of the Σ_t-action rather than the invariants.

In various special cases, our result reduces to more familiar claims. For example, let $\mathcal{R} = H\mathbb{Q}$, the rational Eilenberg-MacLane spectrum, and suppose $\mathcal{O}$ arises from an operad of differential graded rational vector spaces. In the rational context, coinvariants and invariants for a finite group are equivalent so Theorem 1.2 can be thought of as an equivalence between $\mathcal{O}$-algebras and $B(\mathcal{O})$-coalgebras. This is the bar-cobar duality of Ginzburg-Kapranov [47] and Getzler-Jones [46]. For $\mathcal{O}$ the rational associative algebra operad, this is Moore duality [88] between associative algebras and coalgebras. When $\mathcal{O}$ is the rational Lie operad, this is the equivalence between differential graded Lie algebras and differential graded commutative coalgebras that underlies Quillen's work on rational homotopy theory [95], and emerges in the subsequent work of Schlessinger-Stasheff [101] and Sullivan [109]. For a useful historical discussion of operads and Koszul duality phenomena, see Markl-Shnider-Stasheff [79].

Another interesting example is to take $\mathcal{O}$ to be an E_n-operad in $\mathsf{Mod}_{\mathcal{R}}$. Our result amounts to an equivalence between 0-connected non-unital E_n-algebra spectra and $B(E_n)$-coalgebra data; compare with recent work of Lurie [75] and Ayala-Francis [6] who have developed versions of Koszul duality in which E_n-algebras are related to coalgebras over E_n itself. In combination with our result, this is strong evidence for the conjecture that $B(E_n)$ is equivalent to (a desuspended Spanier-Whitehead dual of) E_n itself; compare with Fresse's [43] differential graded E_n operad duality.

Let us describe a further connection between our results and Goodwillie calculus. Kuhn shows in [71] that the layers of the Goodwillie tower of the identity functor for commutative ring spectra can be described in terms of TQ-homology, and uses this description to calculate the $K(n)$-homology of infinite loop-spaces. The same picture holds for algebras over an arbitrary operad $\mathcal{O}$; see also [66, 83, 87, 92]. It follows from this perspective that, when the Goodwillie tower of the identity converges, an $\mathcal{O}$-algebra X can be reconstructed from its TQ-homology. Our paper shows a stronger way this reconstruction can take place through an equivalence of homotopy categories: via a cobar construction on the K-coalgebra $\mathsf{L}Q(X)$ that naturally arises as the spectral algebra analog of Bousfield-Kan completion of a space.

1.5. **Completion of spectral algebras with respect to TQ-homology.** Similar to the context of spaces and ordinary homology, TQ-homology of $\mathcal{O}$-algebras has a comparison map

$$\pi_*(X) \rightarrow \mathsf{TQ}_*(X) \qquad (2)$$

of graded abelian groups, called the Hurewicz map. Assume that X is a cofibrant $\mathcal{O}$-algebra (e.g, let X' be any $\mathcal{O}$-algebra and denote by $X \simeq X'$ its cofibrant replacement). Then (2) comes from a Hurewicz map (Section 3) on the level of $\mathcal{O}$-algebras of the form

$$X \rightarrow UQ(X) \tag{3}$$

in the sense that applying homotopy groups π_* to (3) recovers the map (2).

Once one has such a Hurewicz map on the level of $\mathcal{O}$-algebras, it is natural to form a cosimplicial resolution of X with respect to TQ-homology of the form

$$X \longrightarrow \mathsf{TQ}(X) \rightrightarrows \mathsf{TQ}^2(X) \rightrightarrows \mathsf{TQ}^3(X) \cdots \tag{4}$$

In other words, iterating the point-set level Hurewicz map (3) results in a cosimplicial resolution of X that can be thought of as encoding the spectrum level co-operations on the TQ-homology spectra. The slogan here is that TQ-homology has (point-set level) co-operations and the purpose of the resolution (4) is to encode all of this extra structure. The associated homotopy spectral sequence (Corollary 1.7) is the spectral algebra analog of the unstable Adams spectral sequence of a space; see Bousfield-Kan [20] and the subsequent work of Bendersky-Curtis-Miller [12] and Bendersky-Thompson [13].

Working in the framework of [3] (see also Francis-Gaitsgory [40], Hess [59], and Lurie [75, 6.2]), the homotopy limit of the resolution (4), regarded as a map of cosimplicial $\mathcal{O}$-algebras, naturally occurs as the derived unit map (Definition 5.3) associated to the natural zigzag of weak equivalences (1) comparing $\mathcal{O}$-algebras with coalgebras over the associated comonad K (Proposition 5.2); this is the comonad whose iterations underlie the cosimplicial structure in (4). The right-hand side of the resulting map

$$X \rightarrow X^{\wedge}_{\mathsf{TQ}} \tag{5}$$

is the TQ-homology completion [58] of an $\mathcal{O}$-algebra X (Section 3); this is the precise structured ring spectrum analog of the completions and localizations of spaces originally studied in Sullivan [107, 108], and subsequently in Bousfield-Kan [19] and Hilton-Mislin-Roitberg [60]; there is an extensive literature—for a useful introduction, see also Bousfield [16], Dwyer [32], and May-Ponto [82].

One of the main corollaries of Theorem 1.2 is the following spectral algebra completion result (Corollary 1.6) analogous to several closely related known examples of completion phenomena for a simply-connected pointed space Y: For instance, (i) $Y \simeq Y^{\wedge}_{\mathbb{Z}}$ in Bousfield-Kan [19], (ii) $Y \simeq Y^{\wedge}_{\Omega^n\Sigma^n}$ in Bousfield [17], (iii) $Y \simeq Y^{\wedge}_{\Omega^\infty\Sigma^\infty}$ in Carlsson [21] and subsequently in Arone-Kankaanrinta [4], and (iv) $Y \simeq \operatorname{holim}_\Delta \mathbf{C}(Y,Y)$ where $\mathbf{C}(Y,Y)^n \simeq \Sigma Y \vee \cdots \vee \Sigma Y$ (n-copies) in Hopkins [62].

Corollary 1.6. *With $\mathcal{O}, \mathcal{R}$ as in Theorem 1.2, for any 0-connected $\mathcal{O}$-algebra X, the natural completion map $X \simeq X^{\wedge}_{\mathsf{TQ}}$ is a weak equivalence.*

Associated to the underlying cosimplicial spectrum of the completion $X^{\wedge}_{\mathsf{TQ}}$ is a Bousfield-Kan spectral sequence which plays the role of the unstable Adams spectral sequence. Our results imply that, when X is 0-connected, this spectral sequence converges strongly to the homotopy groups of X (Corollary 1.7).

The following strong convergence result for the TQ-homology completion spectral sequence (Section 8) is a corollary of the connectivity estimates in the proof of Theorem 1.2 (see Theorem 2.7).

Corollary 1.7. *If X is a 0-connected $\mathcal{O}$-algebra, then the* TQ*-homology completion spectral sequence*

$$E^2_{-s,t} = \pi^s\pi_t\mathsf{TQ}^{\cdot}(X) \Longrightarrow \pi_{t-s}X^{\wedge}_{\mathsf{TQ}}$$

converges strongly (Remark 1.8); here, the TQ*-homology cosimplicial resolution* $\mathsf{TQ}^{\cdot}(X)$ *of X denotes a functorial Reedy fibrant replacement of the cosimplicial $\mathcal{O}$-algebra $C(Q(X^c))$, where X^c denotes a functorial cofibrant replacement of X.*

Remark 1.8. By *strong convergence* of $\{E^r\}$ to $\pi_*(X^{\wedge}_{\mathsf{TQ}})$ we mean that (i) for each $(-s,t)$, there exists an r such that $E^r_{-s,t} = E^{\infty}_{-s,t}$ and (ii) for each i, $E^{\infty}_{-s,s+i} = 0$ except for finitely many s; this agrees with [19, 31].

Similarly, the following strong convergence result of the homotopy spectral sequence associated to the cosimplicial cobar construction $C(Y)$ of a K-coalgebra Y follows from the connectivity estimates in the proof of Theorem 1.2 (see Theorem 2.12).

Corollary 1.9. *If Y is a 0-connected cofibrant* K*-coalgebra, then the homotopy spectral sequence*

$$E^2_{-s,t} = \pi^s\pi_t C(Y)^{\mathrm{f}} \Longrightarrow \pi_{t-s}\big(\mathrm{holim}_{\Delta}\, C(Y)\big)$$

converges strongly (Remark 1.8); here, $C(Y)^{\mathrm{f}}$ denotes a functorial Reedy fibrant replacement of the cosimplicial $\mathcal{O}$-algebra $C(Y)$.

Note that Donovan [26] has worked out an interesting algebraic analog of some of the results of this paper for simplicial commutative algebras over a field.

1.10. **Organization of the paper.** We conclude this introduction with an outline of the paper. In Section 2 we outline the argument of our main result. In Section 3 we recall TQ-homology, its calculation by a simplicial bar construction, and setup the cosimplicial cobar constructions associated to TQ-homology completion and the natural zigzag of weak equivalences (1). In Section 4 we setup the homotopy theory of K-coalgebras, which plays a central role in both the statement and proof of our main result. In Section 5 we setup the natural zigzag of weak equivalences (1). Section 6 is where we prove the key homotopical estimates that involve various combinatorial arguments arising from iterated bar construction descriptions of iterated TQ-homology modules. Section 7 is an appendix where we recall some preliminaries on simplicial structures on $\mathcal{O}$-algebras and verify that certain natural transformations associated to TQ-homology mesh well with the simplicial structures. Section 8 is an appendix that develops the needed results on towers of $\mathcal{O}$-algebras associated to the truncation filtration of the simplicial category Δ and the associated Bousfield-Kan homotopy spectral sequence. Section 9 is an appendix to establish that realization commutes with totalization.

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2. Outline of the argument

We will now outline the proof of our main result (Theorem 1.2). It naturally breaks up into five subsidiary results: one for the homotopical analysis of the derived unit map (Definition 5.3), and four associated to the homotopical analysis of the derived counit map (Definition 5.5). As corollaries we also obtain strong convergence results (Corollaries 1.7 and 1.9) for the associated homotopy spectral sequences.

The following relatively weak cofibrancy condition is exploited in the proof of our main result.

Cofibrancy Condition 2.1. *If $\mathcal{O}$ is an operad in $\mathcal{R}$-modules, consider the unit map $\eta\colon I\to\mathcal{O}$ of the operad $\mathcal{O}$ (see* [58, 2.16]*) and assume that $I[r]\to\mathcal{O}[r]$ is a flat stable cofibration (see* [58, 7.7]*) between flat stable cofibrant objects in $\mathsf{Mod}_{\mathcal{R}}$ for each $r\geq 0$.*

The comparison theorem in [58, 3.26, 3.30], which uses a lifting argument that goes back to Rezk [98], shows that the operad $\mathcal{O}$ can be replaced by a weakly equivalent operad $\mathcal{O}'$ satisfying this cofibrancy condition; in fact, it satisfies a much stronger Σ-cofibrancy condition.

2.2. **Homotopical analysis of the TQ-completion map.** Assume that X is a cofibrant $\mathcal{O}$-algebra (Remark 3.2). Then the TQ-homology resolution (4) is precisely the coaugmented cosimplicial $\mathcal{O}$-algebra (Definition 3.14)

$$X\to C(QX)$$

It follows that the TQ-homology completion $X^\wedge_{\mathsf{TQ}}$ of X [58] can be expressed as

$$X^\wedge_{\mathsf{TQ}}\simeq\operatorname{holim}_\Delta C(QX)$$

The following observation is the first step in our attack on analyzing the cosimplicial resolution of an $\mathcal{O}$-algebra with respect to TQ-homology; it points out the homotopical significance of certain cubical diagrams $\mathcal{X}_n$ associated to a coaugmented cosimplicial object. Such diagrams are exploited in [22] and [106] in the contexts of spectra and spaces, respectively, and we use the same argument (see Proposition 6.1).

Proposition 2.3. *Let Z be a cosimplicial $\mathcal{O}$-algebra coaugmented by $d^0\colon Z^{-1}\to Z^0$. If $n\geq 0$, then there are natural zigzags of weak equivalences*

$$\operatorname{hofib}(Z^{-1}\to\operatorname{holim}_{\Delta^{\leq n}}Z)\simeq(\text{iterated hofib})\mathcal{X}_{n+1}$$

where $\mathcal{X}_{n+1}$ denotes the canonical $(n+1)$-cube associated to the coface maps of

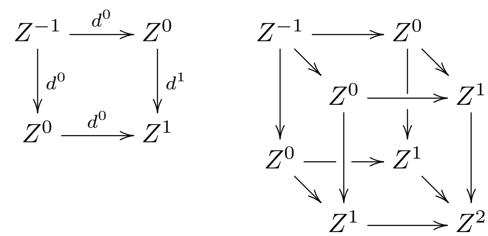
$$Z^{-1}\xrightarrow{d^0}Z^0\overset{d^0}{\underset{d^1}{\rightrightarrows}}Z^1\ \cdots\ Z^n$$

the n-truncation of $Z^{-1}\to Z$. We sometimes refer to $\mathcal{X}_{n+1}$ as the coface $(n+1)$-*cube associated to the coaugmented cosimplicial $\mathcal{O}$-algebra $Z^{-1}\to Z$.*

Remark 2.4. In other words, $\mathcal{X}_{n+1}$ is the canonical $(n+1)$-cube that is built from the coface relations (see [51, I.1])

$$d^j d^i = d^i d^{j-1}, \quad \text{if } i<j,$$

associated to the n-truncation of $Z^{-1}\to Z$. For instance, the coface 2-cube $\mathcal{X}_2$ has the left-hand form



and the coface 3-cube $\mathcal{X}_3$ has the right-hand form where the back 2-cube encodes the relation $d^1d^0 = d^0d^0$, the front 2-cube encodes the relation $d^2d^1 = d^1d^1$, the top 2-cube encodes the relation $d^0d^0 = d^1d^0$, the bottom 2-cube encodes the relation $d^0d^0 = d^1d^0$, the right-hand 2-cube encodes the relation $d^2d^0 = d^0d^1$, and the left-hand 2-cube encodes the relation $d^1d^0 = d^0d^0$. For a useful development, see [89, 106].

We defer the proof of the following homotopical estimates (see Proposition 6.13) to Section 6.

Proposition 2.5. *Let X be a cofibrant $\mathcal{O}$-algebra and $n\geq 1$. Denote by $\mathcal{X}_n$ the coface n-cube associated to the cosimpicial* TQ*-homology resolution $X\to C(QX)$ of X. If X is 0-connected, then the total homotopy fiber of $\mathcal{X}_n$ is n-connected.*

Remark 2.6. It is important to note that the total homotopy fiber of an n-cube of $\mathcal{O}$-algebras is weakly equivalent to its iterated homotopy fiber, and in this paper we use the terms interchangeably; we use the convention that the iterated homotopy fiber of a 0-cube $\mathcal{X}$ (or object $\mathcal{X}_\emptyset$) is the homotopy fiber of the unique map $\mathcal{X}_\emptyset\to *$ and hence is weakly equivalent to $\mathcal{X}_\emptyset$.

Theorem 2.7. *If X is a 0-connected cofibrant $\mathcal{O}$-algebra and $n\geq 0$, then the natural map*

$$X \longrightarrow \operatorname{holim}_{\Delta^{\leq n}} C(QX) \tag{6}$$

is $(n+2)$-connected; for $n=0$ this is the Hurewicz map.

Proof. By Proposition 2.3 the homotopy fiber of the map (6) is weakly equivalent to the total homotopy fiber of the coface $(n+1)$-cube $\mathcal{X}_{n+1}$ associated to $X\to C(QX)$, hence it follows from Proposition 2.5 that the map (6) is $(n+2)$-connected. □

Proof of Theorem 1.2 (first half involving the derived unit map). Assume that X is a cofibrant $\mathcal{O}$-algebra. We want to verify that the natural map $X\to \operatorname{holim}_\Delta C(QX)$ is a weak equivalence. It suffices to verify that the connectivity of the natural map (6) into the n-th stage of the associated holim tower (Section 8) is strictly increasing with n, and Theorem 2.7 completes the proof. □

2.8. **Homotopical analysis of the derived counit map associated to** (1)**.** The following calculation points out the homotopical significance of further cubical diagrams $\mathcal{Y}_n$ associated to a cosimplicial $\mathcal{O}$-algebra. These are exploited in [19, X.6.3] for the Tot tower of a Reedy fibrant cosimplicial pointed space, and the same argument verifies their relevance in our context (see Proposition 6.15).

Proposition 2.9. *Let Z be a cosimplicial $\mathcal{O}$-algebra and $n \geq 0$. There are natural zigzags of weak equivalences*

$$\mathrm{hofib}(\operatorname*{holim}_{\Delta^{\leq n}} Z \to \operatorname*{holim}_{\Delta^{\leq n-1}} Z) \simeq \Omega^n(\text{iterated hofib})\mathcal{Y}_n$$

where $\mathcal{Y}_n$ denotes the canonical n-cube built from the codegeneracy maps of

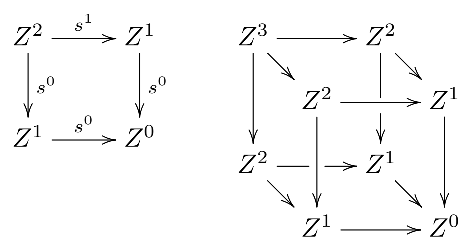
$$Z^0 \xleftarrow{s^0} Z^1 \underset{s^1}{\overset{s^0}{\leftleftarrows}} Z^2 \ \cdots \ Z^n$$

the n-truncation of Z; in particular, $\mathcal{Y}_0$ is the object (or 0-cube) Z^0. Here, Ω^n is weakly equivalent, in the underlying category $\mathsf{Mod}_{\mathcal{R}}$, to the n-fold desuspension Σ^{-n} functor. We often refer to $\mathcal{Y}_n$ as the codegeneracy *n-cube associated to Z.*

Remark 2.10. In other words, $\mathcal{Y}_n$ is the canonical n-cube that is built from the codegeneracy relations (see [51, I.1])

$$s^j s^i = s^i s^{j+1}, \quad \text{if } i \leq j,$$

associated to the n-truncation of Z. For instance, the codegeneracy 2-cube $\mathcal{Y}_2$ has the left-hand form



and the codegeneracy 3-cube $\mathcal{Y}_3$ has the right-hand form where the front 2-cube encodes the relation $s^0s^1 = s^0s^0$, the back 2-cube encodes the relation $s^0s^1 = s^0s^0$, the top 2-cube encodes the relation $s^1s^1 = s^1s^2$, the bottom 2-cube encodes the relation $s^0s^0 = s^0s^1$, the right-hand 2-cube encodes the relation $s^0s^1 = s^0s^0$, and the left-hand 2-cube encodes the relation $s^0s^2 = s^1s^0$. For a useful elaboration, see [89, 106].

The following proposition amounts to calculating homotopical estimates for iterated homotopy fibers (see Proposition 6.21).

Proposition 2.11. *Let Y be a cofibrant K-coalgebra and $n \geq 1$. Denote by $\mathcal{Y}_n$ the codegeneracy n-cube associated to the cosimplicial cobar construction $C(Y)$ of Y. If Y is 0-connected, then the total homotopy fiber of $\mathcal{Y}_n$ is $2n$-connected.*

Theorem 2.12. *If Y is a 0-connected cofibrant K-coalgebra and $n \geq 1$, then the natural map*

$$\operatorname*{holim}_{\Delta^{\leq n}} C(Y) \longrightarrow \operatorname*{holim}_{\Delta^{\leq n-1}} C(Y) \tag{7}$$

is an $(n+1)$-connected map between 0-connected objects.

Proof. The homotopy fiber of the map (7) is weakly equivalent to Ω^n of the total homotopy fiber of the codegeneracy n-cube $\mathcal{Y}_n$ associated to $C(Y)$ by Proposition 2.9, hence by Proposition 2.11 the map (7) is $(n+1)$-connected. □

Theorem 2.13. *If Y is a 0-connected cofibrant K-coalgebra and $n \geq 0$, then the natural maps*

$$\operatorname{holim}_{\Delta} C(Y) \longrightarrow \operatorname{holim}_{\Delta^{\leq n}} C(Y) \tag{8}$$

$$\mathsf{L}Q \operatorname{holim}_{\Delta} C(Y) \longrightarrow \mathsf{L}Q \operatorname{holim}_{\Delta^{\leq n}} C(Y) \tag{9}$$

are $(n+2)$-connected maps between 0-connected objects.

Proof. Consider the first part. By Theorem 2.12 each of the maps in the holim tower $\{\operatorname{holim}_{\Delta^{\leq n}} C(Y)\}_n$ (Section 8), above level n, is at least $(n+2)$-connected. It follows that the map (8) is $(n+2)$-connected. The second part follows from the first part, since by the TQ-Hurewicz theorems (see [58, 1.8, 1.9]), TQ-homology preserves such connectivities. □

We defer the proof of the following theorem (see Theorem 6.32) to Section 6. At the technical heart of the proof lies the spectral algebra higher dual Blakers-Massey theorem [25, 1.11]; these structured ring spectra analogs of Goodwillie's [53] higher cubical diagram theorems were worked out in [25] for the purpose of proving Theorem 2.14 below. The input to the spectral algebra higher dual Blakers-Massey theorem requires the homotopical analysis of an ∞-cartesian $(n+1)$-cube associated to the n-th stage, $\operatorname{holim}_{\Delta^{\leq n}} C(Y)$, of the holim tower associated to $C(Y)$; it is built from coface maps in $C(Y)$. The needed homotopical analysis is worked out (see Theorem 6.32) in Section 6 by leveraging the connectivity estimates in Proposition 2.11 with the fact that codegeneracy maps provide retractions for the appropriate coface maps.

Theorem 2.14. *If Y is a 0-connected cofibrant K-coalgebra and $n \geq 1$, then the natural map*

$$\mathsf{L}Q \operatorname{holim}_{\Delta^{\leq n}} C(Y) \longrightarrow \operatorname{holim}_{\Delta^{\leq n}} \mathsf{L}Q\, C(Y), \tag{10}$$

is $(n+4)$-connected; the map is a weak equivalence for $n = 0$.

The following is a corollary of these connectivity estimates, together with a left cofinality argument in [27, 3.16].

Theorem 2.15. *If Y is a 0-connected cofibrant K-coalgebra, then the natural maps*

$$\mathsf{L}Q \operatorname{holim}_{\Delta} C(Y) \xrightarrow{\simeq} \operatorname{holim}_{\Delta} \mathsf{L}Q\, C(Y) \xrightarrow{\simeq} Y \tag{11}$$

are weak equivalences.

Proof. Consider the left-hand map. It suffices to verify that the connectivities of the natural maps (9) and (10) are strictly increasing with n, and Theorems 2.13 and 2.14 complete the proof. Consider the case of the right-hand map. The right-hand map is a weak equivalence since $\mathsf{L}Q\, C(Y) \simeq \operatorname{Cobar}(\mathsf{K}, \mathsf{K}, Y)$. □

Proof of Theorem 1.2 (second half involving the derived counit map). We want to verify that the natural map

$$\mathsf{L}Q \operatorname{holim}_{\Delta} C(Y) \xrightarrow{\simeq} Y$$

is a weak equivalence; since this is the composite (11), Theorem 2.15 completes the proof. □

3. TQ-homology

In this section we recall the simplicial bar construction associated to a change of operads adjunction. We recall how this plays a role in calculating the derived indecomposables quotient of an $\mathcal{O}$-algebra and how this construction can be modified to produce an iterable point-set model for TQ-homology which plays a key role in the cosimplicial cobar construction arising in the construction of TQ-completion.

3.1. **Simplicial bar constructions.**

Remark 3.2. Unless otherwise noted, we consider $\mathsf{Alg}_{\mathcal{O}}$ equipped with the positive flat stable model structure [58]; one could also work with the positive stable model structure in [38, 58] which has stronger conditions on the cofibrations.

The following is proved in Elmendorf-Mandell [38] for spectral operads arising from operads in simplicial sets; see also [56, 58] for the slight generalization used here.

Proposition 3.3. *Let $f\colon \mathcal{O}\to\mathcal{O}'$ be a map of operads in $\mathcal{R}$-modules. Then the change of operads adjunction*

$$\mathsf{Alg}_{\mathcal{O}} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathsf{Alg}_{\mathcal{O}'} \tag{12}$$

is a Quillen adjunction with left adjoint on top and f^* the forgetful functor (or "restriction along f" functor). If furthermore, f is an objectwise stable equivalence, then (12) *is a Quillen equivalence. Here, $\mathsf{Alg}_{\mathcal{O}}$ and $\mathsf{Alg}_{\mathcal{O}'}$ are equipped with the positive flat stable model structure* [58] *and recall that $f_*(X) = \mathcal{O}'\circ_{\mathcal{O}}(X)$.*

Definition 3.4. Let $\mathcal{O}\to\mathcal{O}'$ be a map of operads in $\mathcal{R}$-modules and X an $\mathcal{O}$-algebra. The *simplicial bar construction* (or two-sided simplicial bar construction) $\mathrm{Bar}(\mathcal{O}',\mathcal{O},X)$ in $(\mathsf{Alg}_{\mathcal{O}'})^{\Delta^{\mathrm{op}}}$ looks like (showing only the face maps)

$$\mathcal{O}'\circ(X) \underset{d_1}{\overset{d_0}{\leftleftarrows}} \mathcal{O}'\circ\mathcal{O}\circ(X) \;\substack{\leftarrow\\ \leftarrow\\ \leftarrow}\; \mathcal{O}'\circ\mathcal{O}\circ\mathcal{O}\circ(X) \;\substack{\leftarrow\\ \leftarrow\\ \leftarrow\\ \leftarrow}\; \cdots$$

and is defined objectwise by $\mathrm{Bar}(\mathcal{O}',\mathcal{O},X)_k := \mathcal{O}'\circ\mathcal{O}^{\circ k}\circ(X)$ with the obvious face and degeneracy maps induced by the multiplication and unit maps [55, A.1], [81, Section 7]; for instance, the indicated coface maps are defined by $d_1 = \mathrm{id}\circ(m)$ where $m\colon \mathcal{O}\circ(X)\to X$ is the left $\mathcal{O}$-action map on X and $d_0 = m\circ\mathrm{id}$ where $m\colon \mathcal{O}'\circ\mathcal{O}\to\mathcal{O}'$ is the right $\mathcal{O}$-action map on $\mathcal{O}'$ given by the composite $\mathcal{O}'\circ\mathcal{O}\to\mathcal{O}'\circ\mathcal{O}'\to\mathcal{O}'$.

Remark 3.5. In more detail, the face maps $d_i\colon \mathrm{Bar}(\mathcal{O}',\mathcal{O},X)_n\to\mathrm{Bar}(\mathcal{O}',\mathcal{O},X)_{n-1}$ are given by

$$\begin{aligned}
d_0&\colon \mathcal{O}'\circ\mathcal{O}\circ\mathcal{O}^{\circ(n-1)}\circ(X)\xrightarrow{m\circ\mathrm{id}^{\circ(n-1)}\circ(\mathrm{id})}\mathcal{O}'\circ\mathcal{O}^{\circ(n-1)}\circ(X), && \text{if } n\geq 1,\\
d_i&\colon \mathcal{O}'\circ\mathcal{O}^{\circ n}\circ(X)\xrightarrow{\mathrm{id}^{\circ i}\circ m\circ\mathrm{id}^{\circ(n-i-1)}\circ(\mathrm{id})}\mathcal{O}'\circ\mathcal{O}^{\circ(n-1)}\circ(X), && \text{if } n\geq 2,\ 1\leq i<n,\\
d_n&\colon \mathcal{O}'\circ\mathcal{O}^{\circ(n-1)}\circ\mathcal{O}\circ(X)\xrightarrow{\mathrm{id}^{\circ n}\circ m}\mathcal{O}'\circ\mathcal{O}^{\circ(n-1)}\circ(X), && \text{if } n\geq 1,
\end{aligned}$$

where $m\colon \mathcal{O}\circ\mathcal{O}\to\mathcal{O}$ is the operad multiplication map on $\mathcal{O}$, and the codegeneracy maps $s_j\colon \mathrm{Bar}(\mathcal{O}',\mathcal{O},X)_n\to\mathrm{Bar}(\mathcal{O}',\mathcal{O},X)_{n+1}$ are given by

$$s_j\colon \mathcal{O}'\circ\mathcal{O}^{\circ j}\circ I\circ\mathcal{O}^{\circ n-j}\circ(X)\xrightarrow{\mathrm{id}^{\circ(j+1)}\circ\eta\circ\mathrm{id}^{\circ(n-j)}\circ(\mathrm{id})}\mathcal{O}'\circ\mathcal{O}^{\circ(n+1)}\circ(X),\quad \text{if } n\geq 0,$$

where I denotes the initial operad and $\eta\colon I\to\mathcal{O}$ denotes the unit map of $\mathcal{O}$.

Proposition 3.6. *Let $f\colon \mathcal{O}\to\mathcal{O}'$ be a morphism of operads in $\mathcal{R}$-modules. Assume that $\mathcal{O}$ satisfies Cofibrancy Condition 2.1. Let X be a cofibrant $\mathcal{O}$-algebra and consider $\mathsf{Alg}_{\mathcal{O}}$ and $\mathsf{Alg}_{\mathcal{O}'}$, with the positive flat stable model structure. Then the natural map*

$$|\operatorname{Bar}(\mathcal{O}',\mathcal{O},X)| \xrightarrow{\simeq} f_*(X) \simeq \mathsf{L}f_*(X)$$

is a weak equivalence; here, $\mathsf{L}f_*$ is the total left derived functor of f_*.

Remark 3.7. This type of calculation in terms of bar constructions goes back to the beginning of operadic time (see, for instance, [37, 80]). Here is one conceptual proof of this calculation. We first use the fact that

$$X \simeq \operatorname*{hocolim}_{\Delta^{\mathrm{op}}} \operatorname{Bar}(\mathcal{O},\mathcal{O},X)$$

in $\mathsf{Alg}_{\mathcal{O}}$ and then apply the total left derived functor $\mathsf{L}f_*$ to get the following

$$\begin{aligned} \mathsf{L}f_*(X) &\simeq \mathsf{L}f_*\Big(\operatorname*{hocolim}_{\Delta^{\mathrm{op}}} \operatorname{Bar}(\mathcal{O},\mathcal{O},X)\Big) \simeq \operatorname*{hocolim}_{\Delta^{\mathrm{op}}} \mathsf{L}f_*\big(\operatorname{Bar}(\mathcal{O},\mathcal{O},X)\big)\\ &\simeq \operatorname*{hocolim}_{\Delta^{\mathrm{op}}} \operatorname{Bar}(\mathcal{O}',\mathcal{O},X) \simeq |\operatorname{Bar}(\mathcal{O}',\mathcal{O},X)| \end{aligned}$$

natural zigzags of weak equivalences; one aim of [57] was to make precise the proofs verifying these weak equivalences; alternately, the original argument exploited in [37] is a reduction to the fact that realization in $\mathcal{O}$-algebras is naturally isomorphic to realization in the underlying category of $\mathcal{R}$-modules equipped with its naturally occurring $\mathcal{O}$-algebra structure [37, VII.3.3]; see also [2, A] and [58, 6.11].

3.8. **Indecomposables of $\mathcal{O}$-algebras.** If X is a non-unital commutative $\mathcal{R}$-algebra spectrum with multiplication map $m\colon X\wedge X\to X$, then the indecomposables quotient X/X^2 is defined by the pushout diagram

$$\begin{array}{ccc} X\wedge X & \xrightarrow{m} & X \\ \downarrow & & \downarrow \\ * & \longrightarrow & X/X^2 \end{array}$$

in the underlying category of $\mathcal{R}$-modules; in other words, it is the $\mathcal{R}$-module built by killing off all 2-ary operations and higher (see Basterra [7]). In particular,

$$X/X^2 \cong X/(X^2, X^3, X^4, \dots)$$

since all 3-ary operations and higher come from iterations of the 2-ary operation m. What about more general flavors of $\mathcal{R}$-algebra spectra, such as an $\mathcal{O}$-algebra X where n-ary operations are parametrized by a map $\mathcal{O}[n] \to \mathbf{hom}_{\mathsf{Mod}_{\mathcal{R}}}(X^{\wedge n}, X)$ of $\mathcal{R}[\Sigma_n]$-modules?

Basterra-Mandell [8, Section 8] construct an indecomposables quotient functor for $\mathcal{R}$-modules equipped with an action of an operad in spaces, via suspension spectra, with trivial 0-ary operations. The basic idea is to kill off all 2-ary operations and higher, similar to earlier work (see, for instance, [41, 73]) in the context of chain complexes. A useful method to determine the correct notion of the indecomposables quotient for operadic spectral algebras, in a formal way, is to use the fact that if X is a non-unital commutative $\mathbb{Z}$-algebra, then the indecomposables quotient X/X^2 is naturally isomorphic to the abelianization [94] of X.

Following this line of inquiry for more general flavors of operadic $\mathbb{Z}$-algebras leads to the following construction of the indecomposables quotient (see, for instance, [41, 42, 58, 72, 73]) with a simple and useful operadic description. Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules such that $\mathcal{O}[0]=*$ and recall [58] that $\tau_1\mathcal{O}$ is defined objectwise by

$$(\tau_1\mathcal{O})[r] := \begin{cases} \mathcal{O}[r], & \text{for } r\leq 1,\\ *, & \text{otherwise,}\end{cases}$$

It follows easily that the natural map $\mathcal{O}{\rightarrow}\tau_1\mathcal{O}$ is a map of operads. The indecomposables quotient functor is given by $\tau_1\mathcal{O}\circ_{\mathcal{O}} -$ for $\mathcal{O}$-algebras, where $\mathcal{O}$ is now any operad of spectra with trivial 0-ary operations. In other words, the indecomposables functor has a very simple and useful description: it is the left adjoint of the change of operads adjunction (12) associated to the canonical map $\mathcal{O}{\rightarrow}\tau_1\mathcal{O}$ of operads.

This construction of the indecomposables quotient functor for $\mathcal{O}$-algebras, in this generality, subsumes the earlier constructions. In particular, the indecomposables quotient functor lands in the category of $\tau_1\mathcal{O}$-algebras, which is isomorphic to the category of left $\mathcal{O}[1]$-modules. There is a twisted group ring structure (see, for instance, [42, 72]) that comes into play when writing down a correct model for the indecomposables of $\mathcal{O}$-algebras, and this structure is precisely encoded by the relative circle product $\tau_1\mathcal{O}\circ_{\mathcal{O}} -$ functor.

Remark 3.9. While many naturally occurring operads $\mathcal{O}$ of spectra can be written as the suspension spectra of operads in spaces, this is not true in general; for instance, endomorphism operads of diagrams [58, 98] in $\mathcal{R}$-modules provide a large family of operads that do not, in general, arise from operads in spaces via suspension spectra; further examples arise in [23, 24].

The topological Quillen homology spectrum of X is, up to weak equivalence, the derived indecomposables quotient of X; but in order to build the TQ-completion [58] of X, we will use a weakly equivalent "fattened-up" version of the indecomposables quotient construction $\tau_1\mathcal{O}\circ_{\mathcal{O}}(X)$ that can be iterated on the point-set (or model category) level to produce iterations of the derived indecomposables quotient functor. We recall this construction now.

3.10. **An iterable point-set model for TQ-homology.** The basic idea in [58] is to replace the category of $\mathcal{O}[1]$-modules, which is isomorphic to the category of $\tau_1\mathcal{O}$-algebras, by a "fattened-up" version so that the forgetful functor back down to $\mathcal{O}$-algebras preserves cofibrant objects; this idea is closely related to, and was motivated by, Carlsson's derived completion construction [22].

Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules such that $\mathcal{O}[0]$ is the null object $*$ and consider any $\mathcal{O}$-algebra X. In order to work with the cosimplicial TQ-homology resolution (4), it will be useful to introduce some notation. Consider any factorization of the canonical map $\mathcal{O}{\rightarrow}\tau_1\mathcal{O}$ in the category of operads as

$$\mathcal{O}\xrightarrow{g} J\xrightarrow{h}\tau_1\mathcal{O}$$

a cofibration followed by a weak equivalence ([58, 3.16]) with respect to the positive flat stable model structure on $\mathsf{Mod}_{\mathcal{R}}$. These maps induce change of operads adjunctions

$$\mathsf{Alg}_{\mathcal{O}} \underset{U}{\overset{Q}{\rightleftarrows}} \mathsf{Alg}_J \underset{h^*}{\overset{h_*}{\rightleftarrows}} \mathsf{Alg}_{\tau_1\mathcal{O}} \cong \mathsf{Mod}_{\mathcal{O}[1]} \tag{13}$$

with left adjoints on top and U, h^* the forgetful functors; here, we denote by $Q := g_*$ the indicated left adjoint and $U := g^*$ the indicated forgetful functor (to simplify notation), where Q is for indecomposable "quotient". It is important to note that since h is a weak equivalence, the right-hand adjunction (h_*, h^*) is a Quillen equivalence (see, for instance, [38, 58]) and hence induces an equivalence on the level of homotopy categories. We think of Alg_J as a "fattened-up" version of $\mathsf{Mod}_{\mathcal{O}[1]}$.

Definition 3.11 (TQ-homology of $\mathcal{O}$-algebras)**.** Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules such that $\mathcal{O}[0] = *$ and X be an $\mathcal{O}$-algebra. The TQ-*homology* $\mathcal{O}$-*algebra* of X is $\mathsf{TQ}(X) := \mathsf{R}U(\mathsf{L}Q(X))$ and the TQ-*homology spectrum* of X underlying $\mathsf{TQ}(X)$ is $\mathsf{L}Q(X)$; if X is furthermore cofibrant in $\mathsf{Alg}_{\mathcal{O}}$, then since the forgetful functor U preserves all weak equivalences, it follows that $\mathsf{TQ}(X) \simeq UQ(X)$ in $\mathsf{Alg}_{\mathcal{O}}$ and $\mathsf{TQ}(X) \simeq Q(X)$ in the underlying category of spectra. Here, $\mathsf{R}U, \mathsf{L}Q$ are the total right and total left derived functors of U, Q, respectively.

Basterra [7] shows that the total left derived functor of indecomposables on non-unital commutative algebra spectra can be calculated by a simplicial bar construction, and since the indecomposables construction is the left adjoint of a change of operads adjunction, it follows (Proposition 3.6) that the TQ-homology of any $\mathcal{O}$-algebra can be calculated by a simplicial bar construction (compare with [42, 83]). This description will be useful in the proof of our main result. It follows that we can calculate the TQ-homology spectrum $\mathsf{L}Q(X)$ of a cofibrant $\mathcal{O}$-algebra X via the simplicial bar constructions

$$\mathsf{L}Q(X) \simeq |\operatorname{Bar}(J, \mathcal{O}, X)| \simeq |\operatorname{Bar}(\tau_1\mathcal{O}, \mathcal{O}, X)|$$

in the underlying category of spectra.

3.12. **Cosimplicial cobar constructions.** Associated to the (Q, U) adjunction in (13) is the monad UQ on $\mathsf{Alg}_{\mathcal{O}}$ and the comonad $\mathsf{K} := QU$ on Alg_J of the form

$$\begin{aligned}
&\mathrm{id} \xrightarrow{\eta} UQ \quad \text{(unit)}, &&\mathrm{id} \xleftarrow{\varepsilon} \mathsf{K} \quad \text{(counit)},\\
&UQUQ \rightarrow UQ \quad \text{(multiplication)}, &&\mathsf{KK} \xleftarrow{m} \mathsf{K} \quad \text{(comultiplication)}.
\end{aligned} \tag{14}$$

It will be useful to interpret the cosimplicial TQ-resolution of X in terms of the following cosimplicial cobar construction involving the comonad K on Alg_J. First note that associated to the adjunction (Q, U) is a right K-coaction $m\colon U \rightarrow U\mathsf{K}$ on U (defined by $m := \eta\mathrm{id}$) and a left K-coaction (or K-coalgebra structure) $m\colon QX \rightarrow \mathsf{K}QX$ on QX (defined by $m = \mathrm{id}\eta\mathrm{id}$), for any $X \in \mathsf{Alg}_{\mathcal{O}}$. In particular, if we restrict to cofibrant objects in $\mathsf{Alg}_{\mathcal{O}}$, then it follows that the TQ-homology spectrum QX underlying the TQ-homology $\mathcal{O}$-algebra $UQ(X)$ of X is naturally equipped with a K-coalgebra structure.

Definition 3.13. Denote by $\eta\colon \mathrm{id} \rightarrow F$ and $m\colon FF \rightarrow F$ the unit and multiplication maps of the simplicial fibrant replacement monad F on Alg_J constructed in Blumberg-Riehl [15, 6.1]; see also Garner [45] and Radulescu-Banu [97].

Definition 3.14. Let Y be an object in $\mathsf{coAlg}_{\mathsf{K}}$. The *cosimplicial cobar constructions* (or two-sided cosimplicial cobar constructions) $C(Y) := \operatorname{Cobar}(U, \mathsf{K}, Y)$ and

$\mathfrak{C}(Y) := \mathrm{Cobar}(U, F\mathsf{K}, FY)$ in $(\mathsf{Alg}_{\mathcal{O}})^{\Delta}$ look like (showing only the coface maps)

$$C(Y): \qquad UY \underset{d^1}{\overset{d^0}{\rightrightarrows}} U\mathsf{K}Y \rightrightarrows\!\!\!\rightarrow U\mathsf{K}\mathsf{K}Y \cdots \tag{15}$$

$$\mathfrak{C}(Y): \qquad UFY \underset{d^1}{\overset{d^0}{\rightrightarrows}} U(F\mathsf{K})FY \rightrightarrows\!\!\!\rightarrow U(F\mathsf{K})(F\mathsf{K})FY \cdots \tag{16}$$

and are defined objectwise by $C(Y)^n := U\mathsf{K}^n Y$ and $\mathfrak{C}(Y)^n := U(F\mathsf{K})^n FY$ with the obvious coface and codegeneracy maps; see, for instance, the face and degeneracy maps in the simplicial bar constructions described in [55, A.1], [81, Section 7], and dualize. For instance, in (15) the indicated coface maps are defined by $d^0 := m\mathrm{id}$ and $d^1 := \mathrm{id}m$, and similarly for (16) where they are modified in the obvious way by insertion of the unit map $\eta\colon \mathrm{id}{\rightarrow}F$ of the simplicial fibrant replacement monad F; these are exactly the kind of resolutions studied in Blumberg-Riehl [15].

Remark 3.15. In more detail, the coface maps $d^i\colon C(Y)^n{\rightarrow}C(Y)^{n+1}$ are given by "insert m at position i", where position is counted from the left starting with 0

$$\begin{aligned}
&d^0\colon U\mathsf{K}^nY \xrightarrow{m\mathrm{id}^{n+1}} U\mathsf{K}^{n+1}Y, && \text{if } n \geq 0,\\
&d^i\colon U\mathsf{K}^nY \xrightarrow{\mathrm{id}^i m \mathrm{id}^{n-i+1}} U\mathsf{K}^{n+1}Y, && \text{if } n \geq 1,\ 1 \leq i \leq n,\\
&d^{n+1}\colon U\mathsf{K}^nY \xrightarrow{\mathrm{id}^{n+1}m} U\mathsf{K}^{n+1}Y, && \text{if } n \geq 0,
\end{aligned}$$

and the codegeneracy maps $s^j\colon C(Y)^n{\rightarrow}C(Y)^{n-1}$ are given by "insert ε at position j", where position is counted from the left starting with 0 at the first copy of K

$$s^j\colon U\mathsf{K}^nY \xrightarrow{\mathrm{id}^{j+1}\varepsilon\mathrm{id}^{n-j}} U\mathsf{K}^{n-1}Y, \quad \text{if } n \geq 1,\ 0 \leq j < n.$$

Similarly, the first and last coface maps $d^0, d^{n+1}\colon \mathfrak{C}(Y)^n{\rightarrow}\mathfrak{C}(Y)^{n+1}$ are given by the composites

$$\begin{aligned}
&d^0\colon U(F\mathsf{K})^nFY \xrightarrow{m\mathrm{id}^{2n+2}} U(\mathrm{id}\mathsf{K})(F\mathsf{K})^nFY \xrightarrow{\mathrm{id}\eta\mathrm{id}^{2n+3}} U(F\mathsf{K})^{n+1}FY, && \text{if } n \geq 0,\\
&d^{n+1}\colon U(F\mathsf{K})^nFY \xrightarrow{\mathrm{id}^{2n+2}m} U(F\mathsf{K})^{n+1}\mathrm{id}Y \xrightarrow{\mathrm{id}^{2n+3}\eta\mathrm{id}} U(F\mathsf{K})^{n+1}FY, && \text{if } n \geq 0,
\end{aligned}$$

and the remaining coface maps $d^i\colon \mathfrak{C}(Y)^n{\rightarrow}\mathfrak{C}(Y)^{n+1}$ are given by the composites

$$d^i\colon U(F\mathsf{K})^nFY \xrightarrow{\mathrm{id}^{2i}m\mathrm{id}^{2n-2i+2}} U(F\mathsf{K})^i(\mathrm{id}\mathsf{K})(F\mathsf{K})^{n-i}FY \xrightarrow{\mathrm{id}^{2i+1}\eta\mathrm{id}^{2n-2i+1}} U(F\mathsf{K})^{n+1}FY$$

if $n \geq 1$, $1 \leq i \leq n$. The codegeneracy maps $s^j\colon \mathfrak{C}(Y)^n{\rightarrow}\mathfrak{C}(Y)^{n-1}$ are given by the composites

$$s^j\colon U(F\mathsf{K})^nFY \xrightarrow{\mathrm{id}^{2j+2}\varepsilon\mathrm{id}^{2n-2j}} U(F\mathsf{K})^j(F\mathrm{id})(F\mathsf{K})^{n-1-j}FY \xrightarrow{\mathrm{id}^{2j+1}m\mathrm{id}^{2n-2j-1}} U(F\mathsf{K})^{n-1}FY$$

if $n \geq 1$, $0 \leq j < n$.

Remark 3.16. It may be helpful to note that while $F\mathsf{K}$ does not inherit the structure of a comonad from K, it does inherit the structure of a non-unital comonad from K; nevertheless, it is easy to verify that the cosimplicial cobar construction (16) is a well-defined cosimplicial $\mathcal{O}$-algebra (see, for instance, [15]).

Proposition 3.17. *Let Y be a K-coalgebra. The unit map $\eta\colon \mathrm{id}{\rightarrow}F$ induces a well-defined natural map of the form $C(Y){\rightarrow}\mathfrak{C}(Y)$ of cosimplicial $\mathcal{O}$-algebras; this map is an objectwise weak equivalence if Y is furthermore a cofibrant K-coalgebra.*

Proof. This is because η is an objectwise acyclic cofibration. □

The reason we introduce the cosimplicial cobar construction $\mathfrak{C}(Y)$, which can be thought of as a "fattened-up" version of $C(Y)$, will become clear in Section 4; it is needed for technical reasons involving the construction of homotopically meaningful mapping spaces of K-coalgebras (i.e., because K may not preserve fibrant objects).

4. Homotopy theory of K-coalgebras

This section is essentially a rehash of [3], with the tiny modification that one can add a simplicial monadic fibrant replacement functor to handle homotopy invariance issues without destroying the cosimplicial structure. It can be thought of as a blending of the ideas in Blumberg-Riehl [15]; in the context of [3] every object is already fibrant and hence no such F was needed.

Recall [77] that a morphism of K-coalgebras from Y to Y' is a map $f\colon Y{\rightarrow}Y'$ in Alg_J that makes the diagram

$$\begin{array}{ccc} Y & \xrightarrow{m} & \mathsf{K}Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{id}f} \\ Y' & \xrightarrow[m]{} & \mathsf{K}Y' \end{array} \tag{17}$$

in Alg_J commute. This motivates the following cosimplicial resolution (18) of the set of K-coalgebra maps from Y to Y'. It will be notationally convenient to introduce the notion of a cofibrant K-coalgebra (defined via the forgetful functor to Alg_J).

Definition 4.1. A morphism in $\mathsf{coAlg}_\mathsf{K}$ is a *cofibration* if the underlying morphism in Alg_J is a cofibration. An object Y in $\mathsf{coAlg}_\mathsf{K}$ is *cofibrant* if the unique map $\emptyset{\rightarrow}Y$ in $\mathsf{coAlg}_\mathsf{K}$ is a cofibration.

Remark 4.2. In $\mathsf{coAlg}_\mathsf{K}$ the initial object $\emptyset$ and the terminal object $*$ are isomorphic; here, the terminal object is the trivial K-coalgebra with underlying object $*$. This follows from the basic assumption that $\mathcal{O}[0] = *$.

Definition 4.3. Let Y, Y' be cofibrant K-coalgebras. The cosimplicial object $\mathbf{Hom}_{\mathsf{Alg}_J}(Y, \mathsf{K}^\bullet Y')$ in sSet looks like (showing only the coface maps)

$$\mathbf{Hom}_{\mathsf{Alg}_J}(Y,Y') \overset{d^0}{\underset{d^1}{\rightrightarrows}} \mathbf{Hom}_{\mathsf{Alg}_J}(Y,\mathsf{K}Y') \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; \mathbf{Hom}_{\mathsf{Alg}_J}(Y,\mathsf{K}\mathsf{K}Y')\cdots \tag{18}$$

and is defined objectwise by $\mathbf{Hom}_{\mathsf{Alg}_J}(Y,\mathsf{K}^\bullet Y')^n := \mathbf{Hom}_{\mathsf{Alg}_J}(Y,\mathsf{K}^n Y')$ with the obvious coface and codegeneracy maps induced by the comultiplication and coaction maps, and counit map, respectively; see, [3, 1.3].

The basic idea is that in simplicial degree 0, the maps d^0, d^1 in (18) send f to the right-hand and left-hand composites in diagram (17), respectively, and that furthermore, the pair of maps d^0, d^1 naturally extend to a cosimplicial diagram; by construction, taking the limit recovers the set of K-coalgebra maps from Y to Y' (Remark 4.8).

But there is a difficulty that arises in this context: $\mathsf{K}^n Y'$ may not be fibrant in Alg_J, and hence (18) is not homotopically meaningful. This is easily corrected by "fattening up" the entries in the codomains without destroying the cosimplicial structure, using a maneuver enabled by, and closely related to, the ideas in Blumberg-Riehl [15]. The following can be thought of as a homotopically meaningful cosimplicial resolution of K-coalgebra maps; it is obtained by appropriately modifying (18) with the simplicial fibrant replacement monad F (Definition 3.13).

Definition 4.4. Let Y, Y' be cofibrant K-coalgebras. The cosimplicial object $\mathbf{Hom}_{\mathsf{Alg}_J}\big(Y, (F\mathsf{K})^\bullet FY'\big)$ in sSet looks like (showing only the coface maps)

$$\mathbf{Hom}_{\mathsf{Alg}_J}(Y, FY') \underset{d^1}{\overset{d^0}{\rightrightarrows}} \mathbf{Hom}_{\mathsf{Alg}_J}\big(Y, (F\mathsf{K})FY'\big) \rightrightarrows\!\!\!\!\rightarrow \mathbf{Hom}_{\mathsf{Alg}_J}\big(Y, (F\mathsf{K})^2 FY'\big) \cdots$$

and is defined objectwise by $\mathbf{Hom}_{\mathsf{Alg}_J}\big(Y, (F\mathsf{K})^\bullet FY'\big)^n := \mathbf{Hom}_{\mathsf{Alg}_J}\big(Y, (F\mathsf{K})^n FY'\big)$ with the obvious coface and codegeneracy maps; compare with [3, 1.3].

These two resolutions can be compared as follows.

Proposition 4.5. *Let Y, Y' be cofibrant K-coalgebras. The unit map $\eta\colon \mathrm{id} \rightarrow F$ induces a well-defined map*

$$\hom_{\mathsf{Alg}_J}(Y, \mathsf{K}^\bullet Y') \longrightarrow \hom_{\mathsf{Alg}_J}\big(Y, (F\mathsf{K})^\bullet FY'\big) \tag{19}$$

$$\textit{resp.}\quad \mathbf{Hom}_{\mathsf{Alg}_J}(Y, \mathsf{K}^\bullet Y') \longrightarrow \mathbf{Hom}_{\mathsf{Alg}_J}\big(Y, (F\mathsf{K})^\bullet FY'\big) \tag{20}$$

of cosimplicial objects in Set *(resp. in* sSet*), natural in all such Y, Y'. Note that* (19) *is obtained from* (20) *by evaluating at simplicial degree* 0.

Proof. This is straightforward. □

Remark 4.6. Recall that if $X, Y \in \mathsf{Alg}_{\mathcal{O}}$, then the mapping space $\mathrm{Map}_{\mathsf{Alg}_{\mathcal{O}}}(X, Y)$ in CGHaus is defined by realization $\mathrm{Map}_{\mathsf{Alg}_{\mathcal{O}}}(X, Y) := |\mathbf{Hom}_{\mathsf{Alg}_{\mathcal{O}}}(X, Y)|$ of the indicated simplicial set.

The following definition of the mapping space of derived K-coalgebra maps appears in [3, 1.10]. Even though the simplicial fibrant replacement monad F was not required in [3] (since every object was already fibrant), the arguments and constructions there easily carry over to this context.

Definition 4.7. Let Y, Y' be cofibrant K-coalgebras. The *mapping spaces* of derived K-coalgebra maps $\mathbf{Hom}_{\mathsf{coAlg}_{\mathsf{K}}}(Y, Y')$ in sSet and $\mathrm{Map}_{\mathsf{coAlg}_{\mathsf{K}}}(Y, Y')$ in CGHaus are defined by the restricted totalizations

$$\mathbf{Hom}_{\mathsf{coAlg}_{\mathsf{K}}}(Y, Y') := \mathrm{Tot}^{\mathsf{res}}\, \mathbf{Hom}_{\mathsf{Alg}_J}\big(Y, (F\mathsf{K})^\bullet FY'\big)$$
$$\mathrm{Map}_{\mathsf{coAlg}_{\mathsf{K}}}(Y, Y') := \mathrm{Tot}^{\mathsf{res}}\, \mathrm{Map}_{\mathsf{Alg}_J}\big(Y, (F\mathsf{K})^\bullet FY'\big)$$

of the indicated cosimplicial objects.

Remark 4.8. To help understand why this is an appropriate notion of the space of derived K-coalgebra maps in our setting, note that there are natural isomorphisms and natural zigzags of weak equivalences

$$\hom_{\mathsf{coAlg}_\mathsf{K}}(Y,Y') \cong \lim_\Delta \hom_{\mathsf{Alg}_J}\bigl(Y,\mathsf{K}^\bullet Y'\bigr),$$
$$\mathbf{Hom}_{\mathsf{coAlg}_\mathsf{K}}(Y,Y') \simeq \operatorname{holim}_\Delta \mathbf{Hom}_{\mathsf{Alg}_J}\bigl(Y,(F\mathsf{K})^\bullet FY'\bigr),$$

respectively.

The following two propositions, pointed out to us by Ogle [90], will be helpful for working both simplicially and topologically with various resolutions; we include a concise proof suggested to us by Dwyer [33] in Section 9.

Proposition 4.9. *If $X \in (\mathsf{sSet})^{\Delta_{\mathsf{res}}}$ is objectwise fibrant and $Y \in (\mathsf{sSet})^{\Delta}$ is Reedy fibrant, then the natural maps*

$$|\operatorname{Tot}^{\mathsf{res}} X| \xrightarrow{\simeq} \operatorname{Tot}^{\mathsf{res}} |X|$$
$$|\operatorname{Tot} Y| \xrightarrow{\simeq} \operatorname{Tot} |Y|$$

in CGHaus *are weak equivalences.*

Proposition 4.10. *If $Y \in (\mathsf{sSet})^{\Delta}$ is objectwise fibrant, then the natural map*

$$|\operatorname{holim}^{\mathsf{BK}}_\Delta Y| \xrightarrow{\simeq} \operatorname{holim}^{\mathsf{BK}}_\Delta |Y| \tag{21}$$

in CGHaus *is a weak equivalence.*

The following is an immediate corollary; it allows us to describe certain maps simplicially and then pass to the topological mapping space via realization.

Proposition 4.11. *Let Y, Y' be cofibrant K-coalgebras. Then the natural map*

$$|\mathbf{Hom}_{\mathsf{coAlg}_\mathsf{K}}(Y,Y')| \xrightarrow{\simeq} \operatorname{Map}_{\mathsf{coAlg}_\mathsf{K}}(Y,Y')$$

is a weak equivalence.

Proof. This follows from Proposition 4.9. □

The following provides a useful language for working with the spaces of derived K-coalgebra maps; compare, [3, 1.11]. The intuition is that cosimplicial degree 0 picks out the underlying map in Alg_J, and the higher cosimplicial degrees encode a "highly homotopy coherent" K-coalgebra structure.

Definition 4.12. Let Y, Y' be cofibrant K-coalgebras. A *derived* K*-coalgebra map* f of the form $Y \rightarrow Y'$ is any map in $(\mathsf{sSet})^{\Delta_{\mathsf{res}}}$ of the form

$$f\colon\ \Delta[-] \longrightarrow \mathbf{Hom}_{\mathsf{Alg}_J}\bigl(Y,(F\mathsf{K})^\bullet FY'\bigr).$$

A *topological derived* K*-coalgebra map* g of the form $Y \rightarrow Y'$ is any map in $(\mathsf{CGHaus})^{\Delta_{\mathsf{res}}}$ of the form

$$g\colon\ \Delta^\bullet \longrightarrow \operatorname{Map}_{\mathsf{Alg}_J}\bigl(Y,(F\mathsf{K})^\bullet FY'\bigr).$$

The *underlying map* of a derived K-coalgebra map f is the map $f_0\colon Y \rightarrow FY'$ that corresponds to the map $f_0\colon \Delta[0] \rightarrow \mathbf{Hom}_{\mathsf{Alg}_J}(Y, FY')$. Note that every derived K-coalgebra map f determines a topological derived K-coalgebra map $|f|$ by realization.

Our next step is to introduce a highly homotopy coherent composition for such derived K-coalgebra maps. The technical tool for making this happen is the box product of cosimplicial objects (below) that can be motivated by consideration of the cup product structure on the singular cochains of a space; a useful introduction and discussion of the box product, from this point of view, is given in McClure-Smith [85, Section 2]—these ideas play a key role in their solution of Deligne's Hochschild cohomology conjecture [84]. The original source for this notion of a monoidal product on cosimplicial objects is Batanin [10, 3.2]; a dual version of the construction appeared earlier in Artin-Mazur [5, III] for bisimplicial sets.

Definition 4.13. If $X, Y \in (\mathsf{sSet})^{\Delta}$ their *box product* $X \square Y \in (\mathsf{sSet})^{\Delta}$ is defined objectwise by a coequalizer of the form

$$(X \square Y)^n \cong \operatorname{colim}\Big(\coprod_{p+q=n} X^p \times Y^q \leftleftarrows \coprod_{r+s=n-1} X^r \times Y^s \Big)$$

where the top (resp. bottom) map is induced by $\mathrm{id} \times d^0$ (resp. $d^{r+1} \times \mathrm{id}$) on each (r,s) term of the indicated coproduct; note that $(X \square Y)^0 \cong X^0 \times Y^0$. The coface maps $d^i\colon (X \square Y)^n \to (X \square Y)^{n+1}$ are induced by

$$\begin{cases} X^p \times Y^q \xrightarrow{d^i \times \mathrm{id}} X^{p+1} \times Y^q, & \text{if } i \leq p, \\ X^p \times Y^q \xrightarrow{\mathrm{id} \times d^{i-p}} X^p \times Y^{q+1}, & \text{if } i > p, \end{cases}$$

and the codegeneracy maps $s^j\colon (X \square Y)^n \to (X \square Y)^{n-1}$ are induced by

$$\begin{cases} X^p \times Y^q \xrightarrow{s^j \times \mathrm{id}} X^{p-1} \times Y^q, & \text{if } j < p, \\ X^p \times Y^q \xrightarrow{\mathrm{id} \times s^{j-p}} X^p \times Y^{q-1}, & \text{if } j \geq p. \end{cases}$$

If $X, Y \in \mathsf{CGHaus}^{\Delta}$, then their box product $X \square Y \in \mathsf{CGHaus}^{\Delta}$ is defined similarly by replacing $(\mathsf{sSet}, \times)$ with $(\mathsf{CGHaus}, \times)$; the box product is defined similarly for cosimplicial objects in any closed symmetric monoidal category $(\mathsf{M}, \otimes)$.

Remark 4.14. For instance, $(X \square Y)^1$ and $(X \square Y)^2$ are naturally isomorphic to the colimits of the left-hand and right-hand diagrams, respectively,

$$\begin{array}{ccc} X^0 \times Y^1 & & \\ {\scriptstyle \mathrm{id}\times d^0}\uparrow & & \\ X^0 \times Y^0 & \xrightarrow{d^1 \times \mathrm{id}} & X^1 \times Y^0 \end{array} \qquad \begin{array}{ccccc} X^0 \times Y^2 & & & & \\ {\scriptstyle \mathrm{id}\times d^0}\uparrow & & & & \\ X^0 \times Y^1 & \xrightarrow{d^1 \times \mathrm{id}} & X^1 \times Y^1 & & \\ & & {\scriptstyle \mathrm{id}\times d^0}\uparrow & & \\ & & X^1 \times Y^0 & \xrightarrow{d^2 \times \mathrm{id}} & X^2 \times Y^0 \end{array}$$

and $(X\square Y)^3$ is naturally isomorphic to the colimit of the diagram

$$\begin{array}{cccccccc}
X^0\times Y^3 & & & & & & \\
\uparrow{\scriptstyle \mathrm{id}\times d^0} & & & & & & \\
X^0\times Y^2 & \xrightarrow{d^1\times\mathrm{id}} & X^1\times Y^2 & & & & \\
 & & \uparrow{\scriptstyle \mathrm{id}\times d^0} & & & & \\
 & & X^1\times Y^1 & \xrightarrow{d^2\times\mathrm{id}} & X^2\times Y^1 & & \\
 & & & & \uparrow{\scriptstyle \mathrm{id}\times d^0} & & \\
 & & & & X^2\times Y^0 & \xrightarrow{d^3\times\mathrm{id}} & X^3\times Y^0
\end{array}$$

The following proposition, proved in McClure-Smith [85, 2.3], provides a conceptual interpretation of the box product. In particular, it provides an alternate description of what it means to give a map out of the box product (Remark 4.16).

Proposition 4.15. *If* $X, Y \in (\mathsf{sSet})^{\Delta}$ *their* box product $X\square Y \in (\mathsf{sSet})^{\Delta}$ *is the left Kan extension of objectwise product along ordinal sum (or concatenation)*

$$\begin{array}{ccccc}
\Delta\times\Delta & \xrightarrow{X\times Y} & \mathsf{sSet}\times\mathsf{sSet} & \xrightarrow{\times} & \mathsf{sSet} \\
\downarrow{\scriptstyle \amalg} & & & & \\
\Delta & & \xrightarrow[\text{left Kan extension}]{X\square Y} & & \mathsf{sSet}
\end{array} \tag{22}$$

Remark 4.16. Let $X, Y, Z \in (\mathsf{sSet})^{\Delta}$. Following the notation in McClure-Smith [85], let's denote the ordinal sum functor by $\Phi\colon\ \Delta\times\Delta{\rightarrow}\Delta$; in other words, $\Phi([r],[s]) = [r]\amalg[s] = [r+s+1]$. Then by (22) there is an isomorphism of hom-sets of the form

$$\hom_{\Delta}(X\square Y, Z) \cong \hom_{\Delta\times\Delta}\big(X\times Y, \Phi^*(Z)\big) \tag{23}$$

natural in X, Y, Z. This correspondence can be described as follows. First note that giving a map of the form $f\colon\ X\square Y{\rightarrow}Z$ is the same as giving a consistent collection [85] of maps of the form

$$f_{p,q}\colon\ X^p\times Y^q{\rightarrow}Z^{p+q}, \qquad p,q\geq 0,$$

and that giving a map of the form $\alpha\colon\ X\times Y{\rightarrow}\Phi^*(Z)$ is the same as giving a consistent collection [85] of maps of the form

$$\alpha_{r,s}\colon\ X^r\times Y^s{\rightarrow}Z^{r+s+1}, \qquad r,s\geq 0.$$

It is straightforward (but tedious) to verify that the natural correspondence (23) is described explicitly by

$$\{f_{p,q}\}\longmapsto\{\alpha_{r,s}\}, \qquad \alpha_{r,s} := (f_{r,s+1})(\mathrm{id}\times d^0) = (f_{r+1,s})(d^{r+1}\times\mathrm{id}) \tag{24}$$

$$\{\alpha_{r,s}\}\longmapsto\{f_{p,q}\}, \qquad f_{p,q} := s^p\alpha_{p,q} \tag{25}$$

where the indicated d^i,s^j denote coface and codegeneracy maps, respectively.

The following can be thought of as analogous to the cup product pairing on the cochain complex of a space. The basic idea is that in cosimplicial degree 0, the composition map μ (below) should send $f\colon Y \to FY'$ and $g\colon Y' \to FY''$ to the composite

$$Y \xrightarrow{f} FY' \xrightarrow{\mathrm{id} g} FFY'' \xrightarrow{m\mathrm{id}} FY''$$

where $m\colon FF \to F$ is the multiplication on F.

Proposition 4.17. *Let Y, Y', Y'' be cofibrant K-coalgebras. There is a natural map of the form*

$$\begin{array}{c} \mathbf{Hom}_{\mathsf{Alg}_J}\big(Y,(F\mathsf{K})^{\bullet}FY'\big)\,\square\,\mathbf{Hom}_{\mathsf{Alg}_J}\big(Y',(F\mathsf{K})^{\bullet}FY''\big) \\ \downarrow{\scriptstyle \mu} \\ \mathbf{Hom}_{\mathsf{Alg}_J}\big(Y,(F\mathsf{K})^{\bullet}FY''\big) \end{array}$$

in $(\mathsf{sSet})^{\Delta}$. *We sometimes refer to μ as the* composition *map.*

Proof. This is proved exactly as in [3, 1.6]; μ is the map induced by the collection of composites

$$\begin{array}{c} \mathbf{Hom}_{\mathsf{Alg}_J}\big(Y,(F\mathsf{K})^{p}FY'\big)\times\mathbf{Hom}_{\mathsf{Alg}_J}\big(Y',(F\mathsf{K})^{q}FY''\big) \\ \downarrow{\scriptstyle \mathrm{id}\times(F\mathsf{K})^pF} \\ \mathbf{Hom}_{\mathsf{Alg}_J}\big(Y,(F\mathsf{K})^{p}FY'\big)\times\mathbf{Hom}_{\mathsf{Alg}_J}\big((F\mathsf{K})^pFY',(F\mathsf{K})^{p}F(F\mathsf{K})^qFY''\big) \\ \downarrow{\scriptstyle \mathrm{comp}} \\ \mathbf{Hom}_{\mathsf{Alg}_J}\big(Y,(F\mathsf{K})^{p}F(F\mathsf{K})^qFY''\big) \\ \downarrow{\scriptstyle \simeq} \\ \mathbf{Hom}_{\mathsf{Alg}_J}\big(Y,(F\mathsf{K})^{p+q}FY''\big) \end{array}$$

where $p, q \geq 0$; here, the indicated weak equivalence is the map induced by multiplication $FF \to F$ of the simplicial fibrant replacement monad. ☐

Proposition 4.18. *Let $A, B \in (\mathsf{sSet})^{\Delta}$. There is a natural isomorphism of the form $|A \square B| \cong |A| \square |B|$ in $(\mathsf{CGHaus})^{\Delta}$.*

Proof. This follows from the fact that realization commutes with finite products and all small colimits [44, 51]. ☐

Proposition 4.19. *Let Y, Y', Y'' be cofibrant K-coalgebras. There is a natural map of the form*

$$\begin{array}{c} \mathrm{Map}_{\mathsf{Alg}_J}\big(Y,(F\mathsf{K})^{\bullet}FY'\big)\,\square\,\mathrm{Map}_{\mathsf{Alg}_J}\big(Y',(F\mathsf{K})^{\bullet}FY''\big) \\ \downarrow{\scriptstyle \mu} \\ \mathrm{Map}_{\mathsf{Alg}_J}\big(Y,(F\mathsf{K})^{\bullet}FY''\big) \end{array}$$

in $(\mathsf{CGHaus})^{\Delta}$. *We sometimes refer to μ as the* composition *map.*

Proof. This follows from Proposition 4.17 by applying realization, together with Proposition 4.18. ☐

The following coaugmentation simply picks out the identity map.

Proposition 4.20. *Let Y be a cofibrant K-coalgebra. There is a coaugmentation map $*\to \mathbf{Hom}_{\mathsf{Alg}_J}(Y,\mathsf{K}^\bullet Y)$ of the form (showing only the coface maps)*

$$* \xrightarrow{d^0} \mathbf{Hom}_{\mathsf{Alg}_J}(Y,Y) \overset{d^0}{\underset{d^1}{\rightrightarrows}} \mathbf{Hom}_{\mathsf{Alg}_J}(Y,\mathsf{K}Y)\cdots$$

Here, the left-hand map picks out the identity map $Y\xrightarrow{\mathrm{id}} Y$ in simplicial degree 0.

Proof. It suffices to observe that $d^0d^0=d^1d^0$. □

Definition 4.21. Let Y be a cofibrant K-coalgebra. The *unit map* ι (compare, [3, 1.6]) is the composite

$$* \to \mathrm{Map}_{\mathsf{Alg}_J}(Y,\mathsf{K}^\bullet Y) \to \mathrm{Map}_{\mathsf{Alg}_J}(Y,(F\mathsf{K})^\bullet FY) \tag{26}$$

in $(\mathsf{CGHaus})^\Delta$. Here, the left-hand map is realization of the coaugmentation in Proposition 4.20, and the right-hand map is realization of the natural map (20).

Definition 4.22. The non-Σ operad A in CGHaus is the coendomorphism operad of $\Delta^\bullet$ with respect to the box product $\square$ ([3, 1.12]) and is defined objectwise by the end construction

$$\mathsf{A}(n) := \mathrm{Map}_{\Delta_{\mathsf{res}}}\big(\Delta^\bullet,(\Delta^\bullet)^{\square n}\big) := \mathrm{Map}\big(\Delta^\bullet,(\Delta^\bullet)^{\square n}\big)^{\Delta_{\mathsf{res}}}$$

In other words, $\mathsf{A}(n)$ is the space of restricted cosimplicial maps from $\Delta^\bullet$ to $(\Delta^\bullet)^{\square n}$; in particular, note that $\mathsf{A}(0)=*$.

Consider the natural collection of maps ([3, 1.13])

$$\begin{aligned}\mathsf{A}(n)\times \mathrm{Map}_{\mathsf{coAlg}_\mathsf{K}}(Y_0,Y_1)\times\cdots\times \mathrm{Map}_{\mathsf{coAlg}_\mathsf{K}}(Y_{n-1},Y_n)\\ \longrightarrow \mathrm{Map}_{\mathsf{coAlg}_\mathsf{K}}(Y_0,Y_n),\qquad n\geq 0,\end{aligned} \tag{27}$$

induced by (iterations of) the composition map μ (Proposition 4.19); in particular, in the case $n=0$, note that (27) denotes the map

$$*=\mathsf{A}(0)\longrightarrow \mathrm{Map}_{\mathsf{coAlg}_\mathsf{K}}(Y_0,Y_0),\qquad n=0,$$

that is $\mathrm{Tot}^{\mathsf{res}}$ applied to the unit map (26).

Remark 4.23. The notion of an A_∞ composition, and the corresponding notion of an A_∞ category, is studied, for instance, in Batanin [11].

Proposition 4.24. *The collection of maps* (27) *determine a topological A_∞ category with objects the cofibrant K-coalgebras and morphism spaces the mapping spaces $\mathrm{Map}_{\mathsf{coAlg}_\mathsf{K}}(Y,Y')$.*

Proof. This is proved exactly as in [3, 1.14]. □

Definition 4.25. The *homotopy category* of K-coalgebras (compare, [3, 1.15]), denoted $\mathsf{Ho}(\mathsf{coAlg}_\mathsf{K})$, is the category with objects the cofibrant K-coalgebras and morphism sets $[X,Y]_\mathsf{K}$ from X to Y the path components

$$[X,Y]_\mathsf{K} := \pi_0\,\mathrm{Map}_{\mathsf{coAlg}_\mathsf{K}}(X,Y)$$

of the indicated mapping spaces.

The next step is to describe the canonical functor from cofibrant K-coalgebras to the homotopy category of K-coalgebras.

Proposition 4.26. *Let X, Y be cofibrant K-coalgebras. There is a natural map of morphism spaces of the form*

$$\hom_{\mathsf{coAlg}_{\mathsf{K}}}(X,Y)\rightarrow \mathrm{Map}_{\mathsf{coAlg}_{\mathsf{K}}}(X,Y) \tag{28}$$

Proof. This is the composite of natural maps

$$\begin{aligned}\hom_{\mathsf{coAlg}_{\mathsf{K}}}(X,Y) &\rightarrow \mathrm{Tot}^{\mathsf{res}}\,\mathrm{Map}_{\mathsf{Alg}_J}(X,\mathsf{K}^\bullet Y)\\ &\rightarrow \mathrm{Tot}^{\mathsf{res}}\,\mathrm{Map}_{\mathsf{Alg}_J}(X,(F\mathsf{K})^\bullet FY) = \mathrm{Map}_{\mathsf{coAlg}_{\mathsf{K}}}(X,Y)\end{aligned}$$

in CGHaus; see [3, 1.14]. □

Proposition 4.27. *Let X, Y be cofibrant K-coalgebras. There is a natural map of morphism sets of the form*

$$\hom_{\mathsf{coAlg}_{\mathsf{K}}}(X,Y)\rightarrow [X,Y]_{\mathsf{K}} \tag{29}$$

Proof. This follows by applying the path components functor π_0 to (28). □

Proposition 4.28. *There is a well-defined functor*

$$\gamma\colon\ \mathsf{coAlg}_K|_{\mathrm{cof}}\rightarrow \mathsf{Ho}(\mathsf{coAlg}_{\mathsf{K}})$$

that is the identity on objects and is the map (29) on morphisms; here, $\mathsf{coAlg}_K|_{\mathrm{cof}} \subset \mathsf{coAlg}_{\mathsf{K}}$ denotes the full subcategory of cofibrant K-coalgebras.

Proof. This is proved exactly as in [3, 1.14]. □

Definition 4.29. A derived K-coalgebra map f of the form $X\rightarrow Y$ is a *weak equivalence* if the underlying map $f_0\colon\ X\rightarrow FY$ is a weak equivalence.

The following verifies that this definition is homotopically meaningful.

Proposition 4.30. *Let X, Y be cofibrant K-coalgebras. A derived K-coalgebra map f of the form $X\rightarrow Y$ is a weak equivalence if and only if the induced map $\gamma(f)$ in $[X,Y]_{\mathsf{K}}$ is an isomorphism in the homotopy category of K-coalgebras.*

Proof. This is proved exactly as in [3, 1.16]. □

5. The derived unit and counit maps

The purpose of this section is to setup the natural zigzag of weak equivalences (Proposition 5.2) in (1) that naturally arises in the framework of Section 4 and to describe the associated derived unit maps (Definition 5.3) and derived counit maps (Definition 5.5).

We first observe that the adjunction (13) meshes nicely with mapping spaces.

Proposition 5.1. *Let $X \in \mathsf{Alg}_{\mathcal{O}}$ and $Y \in \mathsf{coAlg}_{\mathsf{K}}$. The natural isomorphisms associated to the (Q,U) adjunction (13) induce well-defined isomorphisms*

$$\mathbf{Hom}_{\mathsf{Alg}_J}(QX,\mathsf{K}^\bullet Y) \xrightarrow{\cong} \mathbf{Hom}_{\mathsf{Alg}_{\mathcal{O}}}(X,U\mathsf{K}^\bullet Y)$$

$$\mathbf{Hom}_{\mathsf{Alg}_J}\big(QX,(F\mathsf{K})^\bullet FY\big) \xrightarrow{\cong} \mathbf{Hom}_{\mathsf{Alg}_{\mathcal{O}}}\big(X,U(F\mathsf{K})^\bullet FY\big)$$

of cosimplicial objects in sSet, natural in X, Y.

Proof. The first case is because the composite

$$\hom(Q(X)\dot{\otimes}\Delta[n],\mathsf{K}^\bullet Y) \cong \hom(Q(X\dot{\otimes}\Delta[n]),\mathsf{K}^\bullet Y) \cong \hom(X\dot{\otimes}\Delta[n],U\mathsf{K}^\bullet Y)$$

is a well-defined map of cosimplicial objects in Set, natural in X, Y, for each $n \geq 0$; the second case involving the simplicial fibrant replacement monad F is similar. □

The desired natural zigzags of weak equivalences now follows using totalization models for the homotopy limits.

Proposition 5.2. *Let X be a cofibrant $\mathcal{O}$-algebra and Y a cofibrant K-coalgebra. There are natural zigzags of weak equivalences of the form*

$$\mathrm{Map}_{\mathsf{coAlg}_{\mathsf{K}}}(QX,Y) \simeq \mathrm{Map}_{\mathsf{Alg}_{\mathcal{O}}}\big(X, \mathrm{holim}_\Delta C(Y)\big)$$

in CGHaus*; applying π_0 gives the natural isomorphism* $[QX,Y]_{\mathsf{K}} \cong [X, \mathrm{holim}_\Delta C(Y)]$.

Proof. There are natural zigzags of weak equivalences of the form

$$\begin{aligned}\mathbf{Hom}_{\mathsf{Alg}_{\mathcal{O}}}(X, \mathrm{holim}_\Delta C(Y)) &\simeq \mathbf{Hom}_{\mathsf{Alg}_{\mathcal{O}}}\big(X, \mathrm{Tot}^{\mathsf{res}}\,\mathfrak{C}(Y)\big)\\ &\cong \mathrm{Tot}^{\mathsf{res}}\,\mathbf{Hom}_{\mathsf{Alg}_{\mathcal{O}}}\big(X, U(F\mathsf{K})^\bullet FY\big)\\ &\cong \mathrm{Tot}^{\mathsf{res}}\,\mathbf{Hom}_{\mathsf{Alg}_J}\big(QX, (F\mathsf{K})^\bullet FY\big)\\ &\cong \mathbf{Hom}_{\mathsf{coAlg}_{\mathsf{K}}}(QX,Y)\end{aligned}$$

in sSet; applying realization finishes the proof. □

These natural zigzags of weak equivalences induce an adjunction on the level of homotopy categories. It will be sufficient to describe explicit models for the underlying unit and counit maps; we call these the derived unit and derived counit maps, respectively.

Definition 5.3. Let X be a cofibrant $\mathcal{O}$-algebra (Remark 3.2). The *derived unit map* associated to the natural zigzag of weak equivalences in Proposition 5.2 is the $\mathcal{O}$-algebra map of the form $X \longrightarrow \mathrm{holim}_\Delta C(QX)$ with representing map

$$X \longrightarrow \mathrm{Tot}^{\mathsf{res}}\,\mathfrak{C}(QX) \tag{30}$$

corresponding to the identity map $\mathrm{id}\colon Q(X) {\rightarrow} Q(X)$ in $\mathsf{coAlg}_{\mathsf{K}}$.

The basic idea behind Definition 5.5 is to look for a naturally occurring derived K-coalgebra map of the form (31); the construction follows immediately from the observation that there are natural zigzags of weak equivalences

$$\operatorname*{holim}_\Delta C(Y) \simeq \operatorname*{holim}_\Delta \mathfrak{C}(Y) \simeq \mathrm{Tot}^{\mathsf{res}}\,\mathfrak{C}(Y)$$

of $\mathcal{O}$-algebras.

Definition 5.4. Denote by c the simplicial cofibrant replacement functor on $\mathsf{Alg}_{\mathcal{O}}$ ([39, I.C.11], [99, 6.3]).

Definition 5.5. Let Y be a cofibrant K-coalgebra (Definitions 4.1 and 3.2). The *derived counit map* associated to the natural zigzag of weak equivalences in Proposition 5.2 is the derived K-coalgebra map of the form

$$\mathsf{L}Q \operatorname*{holim}_\Delta C(Y) \rightarrow Y \tag{31}$$

with underlying map

$$Qc\,\mathrm{Tot}^{\mathsf{res}}\,\mathfrak{C}(Y) \rightarrow Y \tag{32}$$

corresponding to the "fattened-up" version of the identity map given by

$$c\,\mathrm{Tot}^{\mathsf{res}}\,\mathfrak{C}(Y) \xrightarrow{\simeq} \mathrm{id}\,\mathrm{Tot}^{\mathsf{res}}\,\mathfrak{C}(Y) \tag{33}$$

in $\mathsf{Alg}_{\mathcal{O}}$, via the adjunctions (49) and (46). More precisely, the derived counit map is the derived K-coalgebra map defined by the composite

$$\begin{aligned}\Delta[-] &\xrightarrow{(*)} \mathbf{Hom}_{\mathsf{Alg}_{\mathcal{O}}}\big(c\,\mathrm{Tot}^{\mathsf{res}}\,\mathfrak{C}(Y), \mathfrak{C}(Y)\big)\\ &\cong\ \mathbf{Hom}_{\mathsf{Alg}_J}\big(Qc\,\mathrm{Tot}^{\mathsf{res}}\,\mathfrak{C}(Y), (F\mathsf{K})^{\bullet}FY\big)\end{aligned} \tag{34}$$

in $(\mathsf{sSet})^{\Delta_{\mathsf{res}}}$, where $(*)$ corresponds to the map (33).

Remark 5.6. Note that if X is a cofibrant $\mathcal{O}$-algebra, then there is a zigzag of weak equivalences of the form

$$X^{\wedge}_{\mathsf{TQ}} \simeq\ \mathrm{holim}_{\Delta}\, C(QX) \simeq\ \mathrm{Tot}^{\mathsf{res}}\,\mathfrak{C}(QX) \tag{35}$$

in $\mathsf{Alg}_{\mathcal{O}}$, natural with respect to all such X; this is because $\mathfrak{C}(QX)$ is objectwise fibrant.

The following homotopically fully faithful property was pointed out in [59, 5.5], and subsequently in [3, 2.15].

Remark 5.7. Let X, X' be cofibrant $\mathcal{O}$-algebras. If X is fibrant and the natural coaugmentation $X' \simeq X'^{\wedge}_{\mathsf{TQ}}$ is a weak equivalence, then there is a natural zigzag

$$Q\colon\ \mathrm{Map}_{\mathsf{Alg}_{\mathcal{O}}}(X, X') \xrightarrow{\simeq} \mathrm{Map}_{\mathsf{coAlg}_{\mathsf{K}}}(QX, QX')$$

of weak equivalences; applying π_0 gives the map $[f] \mapsto [Qf]$. This follows from the natural zigzags

$$\begin{aligned}\mathrm{Map}_{\mathsf{Alg}_{\mathcal{O}}}(X, X') &\simeq\ \mathrm{Map}_{\mathsf{Alg}_{\mathcal{O}}}(X, X'^{\wedge}_{\mathsf{TQ}})\\ &\simeq\ \mathrm{Map}_{\mathsf{Alg}_{\mathcal{O}}}\big(X, \mathrm{holim}_{\Delta}\, C(QX')\big) \simeq\ \mathrm{Map}_{\mathsf{coAlg}_{\mathsf{K}}}(QX, QX')\end{aligned}$$

of weak equivalences; see [3, 2.15].

6. Homotopical analysis of cubical diagrams

The purpose of this section is to prove Propositions 2.3, 2.5, 2.9, 2.11, and Theorem 2.14 that were needed in the proof of our main result (Section 2). Aspects of our approach are in the same spirit as the work of Dundas [29], where Goodwillie's higher Blakers-Massey theorems [53] for spaces are exploited to great effect.

Proposition 6.1 (Proposition 2.3 restated)**.** *Let Z be a cosimplicial $\mathcal{O}$-algebra coaugmented by $d^0\colon Z^{-1} \to Z^0$. If $n \geq 0$, then there are natural zigzags of weak equivalences*

$$\mathrm{hofib}(Z^{-1} \to \mathrm{holim}_{\Delta^{\leq n}}\, Z) \simeq (\mathrm{iterated\,hofib})\mathcal{X}_{n+1}$$

where $\mathcal{X}_{n+1}$ denotes the canonical $(n+1)$-cube associated to the coface maps of

$$Z^{-1} \xrightarrow{d^0} Z^0 \underset{d^1}{\overset{d^0}{\rightrightarrows}} Z^1\ \cdots\ Z^n$$

the n-truncation of $Z^{-1} \to Z$. We sometimes refer to $\mathcal{X}_{n+1}$ as the coface $(n+1)$*-cube associated to the coaugmented cosimplicial $\mathcal{O}$-algebra $Z^{-1} \to Z$.*

Proof. This follows easily from left cofinality (Propositions 8.27 and 8.30) together with the fact that the iterated hofiber of a cubical diagram is weakly equivalent to its total hofiber. A proof is elaborated in [89, 9.4.5] and [106]. ∎

Remark 6.2. For instance, in the case $n = 1$, consider the left-hand coface 2-cube $\mathcal{X}_2$ of the form

$$\begin{array}{ccc} Z^{-1} & \xrightarrow{d^0} & Z^0 \\ \downarrow{\scriptstyle d^0} & & \downarrow{\scriptstyle d^0} \\ Z^0 & \xrightarrow{d^1} & Z^1 \end{array} \qquad\qquad \begin{array}{ccc} & & Z^0 \\ & & \downarrow{\scriptstyle d^0} \\ Z^0 & \xrightarrow{d^1} & Z^1 \end{array}$$

The homotopy limit of the right-hand punctured 2-cube is weakly equivalent (Proposition 8.30) to $\mathrm{holim}_{\Delta^{\leq 1}} Z$. It follows that the homotopy fiber of the induced map $Z^{-1} \to \mathrm{holim}_{\Delta^{\leq 1}} Z$ is weakly equivalent to the iterated homotopy fiber of $\mathcal{X}_2$.

The following homotopy spectral sequence for a simplicial symmetric spectrum is well known; for a recent reference, see [37, X.2.9] and [64, 4.3]. It will be useful for estimating connectivities below.

Proposition 6.3. *Let Y be a simplicial symmetric spectrum. There is a natural homologically graded spectral sequence in the right-half plane such that*

$$E^2_{p,q} = H_p(\pi_q(Y)) \Longrightarrow \pi_{p+q}(|Y|)$$

Here, $\pi_q(Y)$ denotes the simplicial abelian group obtained by applying π_q levelwise to Y.

The following calculations, which use the notation in [58, Section 2], encode all of the combinatorics needed for calculating explicitly the iterated homotopy fibers of the coface and codegeneracy n-cubes discussed below. In particular, these combinatorics allow us to efficiently calculate connectivities by using exactly the same line of arguments that were used in [58, Proof of 1.8] for calculating the homotopy fibers of certain 1-cubes, but now iterated up to calculate the total homotopy fibers of certain n-cubes.

Proposition 6.4. *Consider symmetric sequences in $\mathsf{Mod}_{\mathcal{R}}$. Let $Y \in \mathsf{SymSeq}$ such that $Y[0] = *$. If $n \geq 1$, then the diagram*

$$\begin{array}{ccc} (Y^{\check{\otimes} n})^{>n} & \xrightarrow{\subset} & Y^{\check{\otimes} n} \\ \downarrow & & \downarrow{\scriptstyle (*)} \\ * & \longrightarrow & (\tau_1 Y)^{\check{\otimes} n} \end{array}$$

is a pushout diagram, and a pullback diagram, in SymSeq. In particular, the fiber of the map $(*)$ is a symmetric sequence that starts at level $n+1$.

Proof. This is straightforward. □

Proposition 6.5. *Consider symmetric sequences in $\mathsf{Mod}_{\mathcal{R}}$. Let $W, Z \in \mathsf{SymSeq}$ such that $W[0] = *$ and $Z[0] = *$. If $m \geq 0$, then $W^{>m} \circ Z = (W^{>m} \circ Z)^{>m}$. In other words, if W starts at level $m+1$, then $W \circ Z$ starts at level $m+1$.*

Proof. This is immediate. □

Proposition 6.6. *Consider symmetric sequences in* $\mathsf{Mod}_{\mathcal{R}}$. *Let* $W, Y \in \mathsf{SymSeq}$ *such that* $W[0] = *$ *and* $Y[0] = *$. *If* $m \geq 0$, *then the left-hand diagram*

$$\begin{array}{ccc} \widetilde{W} = \widetilde{W}^{>m+1} & \xrightarrow{\subset} & W^{>m} \circ Y \\ \downarrow & & \downarrow (*) \\ * & \longrightarrow & W^{>m} \circ \tau_1 Y \end{array} \qquad \widetilde{W} := \coprod_{t > m} W[t] \wedge_{\Sigma_t} (Y^{\check{\otimes} t})^{>t}$$

is a pushout diagram, and a pullback diagram, in SymSeq. *In particular, the fiber of the map* $(*)$ *is a symmetric sequence that starts at level* $m+2$.

Proof. This is straightforward. □

Definition 6.7. Let $n \geq 1$ and denote by Cube_n the category with objects the vertices $(v_1, \dots, v_n) \in \{0,1\}^n$ of the unit n-cube. There is at most one morphism between any two objects, and there is a morphism $(v_1, \dots, v_n) \to (v_1', \dots, v_n')$ if and only if $v_i \leq v_i'$ for each $1 \leq i \leq n$. In particular, Cube_n is the category associated to a partial order on the set $\{0,1\}^n$.

For each $n \geq 1$, denote by $\mathcal{X}_n$ the coface n-cube associated to the coaugmentation $X \to C(QX)$. The reason for introducing the n-cubes $\mathcal{X}_n'$, $\mathcal{X}_n''$, $\mathcal{X}_n'''$ in Definition 6.8 below is that we can homotopically analyze the iterated homotopy fibers of $\mathcal{X}_n'''$ using the combinatorics developed above; since these intermediate n-cubes fit into a zigzag of weak equivalences $\mathcal{X}_n \simeq \mathcal{X}_n'''$ of n-cubes, the combinatorial analysis results in a homotopical analysis of $\mathcal{X}_n$ (Proposition 6.10).

Definition 6.8. If $X \in \mathsf{Alg}_{\mathcal{O}}$ is cofibrant and $n \geq 1$, denote by $\mathcal{X}_n$ the coface n-cube associated to $X \to C(QX)$, and by $\mathcal{X}_n'$, $\mathcal{X}_n''$, $\mathcal{X}_n'''$ the following weakly equivalent n-cubes. The functor $\mathcal{X}_n' \colon \mathsf{Cube}_n \to \mathsf{Alg}_{\mathcal{O}}$ is defined by

$$(v_1, v_2, \dots, v_n) \longmapsto A_1 \circ_{\mathcal{O}} A_2 \circ_{\mathcal{O}} \cdots A_n \circ_{\mathcal{O}} (X)$$
$$\text{where} \qquad A_i := \begin{cases} \mathcal{O}, & \text{for } v_i = 0, \\ J, & \text{for } v_i = 1, \end{cases}$$

with maps induced by $\mathcal{O} \to J$; the functor $\mathcal{X}_n'' \colon \mathsf{Cube}_n \to \mathsf{Alg}_{\mathcal{O}}$ is defined by

$$(v_1, v_2, \dots, v_n) \longmapsto |\operatorname{Bar}(A_1, \mathcal{O}, |\operatorname{Bar}(A_2, \mathcal{O}, \dots, |\operatorname{Bar}(A_n, \mathcal{O}, X)| \cdots)|)|$$
$$\text{where} \qquad A_i := \begin{cases} \mathcal{O}, & \text{for } v_i = 0, \\ J, & \text{for } v_i = 1, \end{cases}$$

with maps induced by $\mathcal{O} \to J$; the functor $\mathcal{X}_n''' \colon \mathsf{Cube}_n \to \mathsf{Alg}_{\mathcal{O}}$ is defined by

$$(v_1, v_2, \dots, v_n) \longmapsto |\operatorname{Bar}(A_1, \mathcal{O}, |\operatorname{Bar}(A_2, \mathcal{O}, \dots, |\operatorname{Bar}(A_n, \mathcal{O}, X)| \cdots)|)|$$
$$\text{where} \qquad A_i := \begin{cases} \mathcal{O}, & \text{for } v_i = 0, \\ \tau_1 \mathcal{O}, & \text{for } v_i = 1, \end{cases}$$

with maps induced by $\mathcal{O} \to \tau_1 \mathcal{O}$.

Remark 6.9. For instance, $\mathcal{X}_2$ is the 2-cube associated to $X \to C(QX)$ of the form

$$\begin{array}{ccc} X & \xrightarrow{d^0} & J \circ_{\mathcal{O}} (X) \\ \downarrow d^0 & & \downarrow d^0 \\ J \circ_{\mathcal{O}} (X) & \xrightarrow{d^1} & J \circ_{\mathcal{O}} J \circ_{\mathcal{O}} (X) \end{array}$$

$\mathcal{X}'_2$ is the associated 2-cube of the form

$$\begin{array}{ccc} \mathcal{O}\circ_{\mathcal{O}}\mathcal{O}\circ_{\mathcal{O}}(X) & \xrightarrow{d^0} & J\circ_{\mathcal{O}}\mathcal{O}\circ_{\mathcal{O}}(X) \\ \downarrow{\scriptstyle d^0} & & \downarrow{\scriptstyle d^0} \\ \mathcal{O}\circ_{\mathcal{O}}J\circ_{\mathcal{O}}(X) & \xrightarrow{d^1} & J\circ_{\mathcal{O}}J\circ_{\mathcal{O}}(X) \end{array}$$

$\mathcal{X}''_2$ is the associated 2-cube of the form

$$\begin{array}{ccc} |\operatorname{Bar}(\mathcal{O},\mathcal{O},|\operatorname{Bar}(\mathcal{O},\mathcal{O},X)|)| & \xrightarrow{d^0} & |\operatorname{Bar}(J,\mathcal{O},|\operatorname{Bar}(\mathcal{O},\mathcal{O},X)|)| \\ \downarrow{\scriptstyle d^0} & & \downarrow{\scriptstyle d^0} \\ |\operatorname{Bar}(\mathcal{O},\mathcal{O},|\operatorname{Bar}(J,\mathcal{O},X)|)| & \xrightarrow{d^1} & |\operatorname{Bar}(J,\mathcal{O},|\operatorname{Bar}(J,\mathcal{O},X)|)| \end{array}$$

and $\mathcal{X}'''_2$ is the associated 2-cube of the form

$$\begin{array}{ccc} |\operatorname{Bar}(\mathcal{O},\mathcal{O},|\operatorname{Bar}(\mathcal{O},\mathcal{O},X)|)| & \xrightarrow{d^0} & |\operatorname{Bar}(\tau_1\mathcal{O},\mathcal{O},|\operatorname{Bar}(\mathcal{O},\mathcal{O},X)|)| \\ \downarrow{\scriptstyle d^0} & & \downarrow{\scriptstyle d^0} \\ |\operatorname{Bar}(\mathcal{O},\mathcal{O},|\operatorname{Bar}(\tau_1\mathcal{O},\mathcal{O},X)|)| & \xrightarrow{d^1} & |\operatorname{Bar}(\tau_1\mathcal{O},\mathcal{O},|\operatorname{Bar}(\tau_1\mathcal{O},\mathcal{O},X)|)| \end{array} \tag{36}$$

Proposition 6.10. *If X is a cofibrant $\mathcal{O}$-algebra and $n\geq 1$, then the natural maps*

$$\mathcal{X}_n \cong \mathcal{X}'_n \xleftarrow{\simeq} \mathcal{X}''_n \xrightarrow{\simeq} \mathcal{X}'''_n \tag{37}$$

of n-cubes are objectwise weak equivalences. If furthermore, X is 0-connected, then the iterated homotopy fiber of $\mathcal{X}'''_n$ is n-connected, and hence the total homotopy fiber of $\mathcal{X}_n$ is n-connected.

Proof. The case $n=1$ is proved in [58, 1.8]. By using exactly the same arguments as in [58, 1.8], but for the objectwise calculation of the iterated fibers of the corresponding n-multisimplicial objects, it follows easily but tediously from the combinatorics in Propositions 6.4, 6.5, and 6.6 that the iterated homotopy fiber is weakly equivalent to the realization of a simplicial object which is n-connected in each simplicial degree; by Proposition 6.3 the realization of such an object is n-connected. To finish off the proof, it suffices to verify the maps of cubes are objectwise weak equivalences; this follows from Proposition 6.12 below. □

Remark 6.11. For instance, consider the case $n=2$. Removing the outer realization in diagram (36), evaluating at (horizontal) simplicial degree r, and calculating the fibers, together with the induced map $(*)$, we obtain the commutative diagram

$$\begin{array}{ccccc} \mathcal{O}^{>1}\circ\mathcal{O}^{\circ r}\circ Z' & \xrightarrow{\subset} & \mathcal{O}\circ\mathcal{O}^{\circ r}\circ Z' & \longrightarrow & \tau_1\mathcal{O}\circ\mathcal{O}^{\circ r}\circ Z' \\ \downarrow{\scriptstyle (*)} & & \downarrow & & \downarrow \\ \mathcal{O}^{>1}\circ\mathcal{O}^{\circ r}\circ Z & \xrightarrow{\subset} & \mathcal{O}\circ\mathcal{O}^{\circ r}\circ Z & \longrightarrow & \tau_1\mathcal{O}\circ\mathcal{O}^{\circ r}\circ Z \end{array} \tag{38}$$

where $Z' := |\operatorname{Bar}(\mathcal{O},\mathcal{O},X)|$, $Z := |\operatorname{Bar}(\tau_1\mathcal{O},\mathcal{O},X)|$, and the rows are cofiber sequences in $\mathsf{Mod}_{\mathcal{R}}$. We want to calculate the fiber of the map $(*)$. Removing the realization from $(*)$ in diagram (38) and evaluating at (vertical) simplicial degree s, we obtain the commutative diagram

$$\begin{array}{ccccc} (\text{iterated hofib})_{r,s} = \big(\widetilde{W(r)}\big)^{>2} \circ \mathcal{O}^{\circ s} \circ X & \xrightarrow{\subset} & W(r) \circ \mathcal{O} \circ \mathcal{O}^{\circ s} \circ X \\ \downarrow & & \downarrow \\ * & \longrightarrow & W(r) \circ \tau_1 \mathcal{O} \circ \mathcal{O}^{\circ s} \circ X \end{array}$$

where

$$W(r) := \mathcal{O}^{>1} \circ \mathcal{O}^{\circ r}$$
$$\widetilde{W(r)} := \coprod_{t>1} W(r)[t] \wedge_{\Sigma_t} (\mathcal{O}^{\check{\otimes} t})^{>t}$$

The indicated square is a pushout diagram in $\mathsf{Mod}_{\mathcal{R}}$ and a pullback diagram in $\mathsf{Alg}_{\mathcal{O}}$; here, we used Propositions 6.5 and 6.6. Since X is 0-connected, it follows that $(\text{iterated hofib})_{r,s}$ is 2-connected for each $r, s \geq 0$. Hence, applying realization in the vertical direction gives a (horizontal) simplicial $\mathcal{O}$-algebra that is 2-connected in every simplicial degree r, and therefore realization in the horizontal direction gives an $\mathcal{O}$-algebra that is 2-connected; hence the total homotopy fiber of $\mathcal{X}_2'''$ is 2-connected.

The following is proved in [91, 93] and is closely related to similar statements in [37, 105]; it implies that the maps of cubes in (37) are objectwise weak equivalences (Proposition 3.6).

Proposition 6.12. *Let $f\colon \mathcal{O} \rightarrow \mathcal{O}'$ be a morphism of operads in $\mathcal{R}$-modules such that $\mathcal{O}[0] = *$. Assume that $\mathcal{O}$ satisfies Cofibrancy Condition 2.1. Let Y be an $\mathcal{O}$-algebra and consider the simplicial bar construction* $\mathrm{Bar}(\mathcal{O}', \mathcal{O}, Y)$.

- (a) *If $j\colon A \rightarrow B$ is a cofibration between cofibrant objects in $\mathsf{Alg}_{\mathcal{O}}$, then j is a positive flat stable cofibration in $\mathsf{Mod}_{\mathcal{R}}$.*
- (b) *If A is a cofibrant $\mathcal{O}$-algebra, then A is positive flat stable cofibrant in $\mathsf{Mod}_{\mathcal{R}}$.*
- (c) *If Y is positive flat stable cofibrant in $\mathsf{Mod}_{\mathcal{R}}$, then $|\mathrm{Bar}(\mathcal{O}', \mathcal{O}, Y)|$ is cofibrant in $\mathsf{Alg}_{\mathcal{O}'}$.*

The following proposition gives the connectivity estimates that we need.

Proposition 6.13 (Proposition 2.5 restated)**.** *Let X be a cofibrant $\mathcal{O}$-algebra and $n \geq 1$. Denote by $\mathcal{X}_n$ the coface n-cube associated to the cosimpicial TQ-homology resolution $X \rightarrow C(QX)$ of X. If X is 0-connected, then the total homotopy fiber of $\mathcal{X}_n$ is n-connected.*

Proof. This is a special case of Proposition 6.10. □

The following observation in [89, 3.4.8] provides a useful building block for proving, with minimal effort, several useful propositions below.

Proposition 6.14. *Consider any 2-cube $\mathcal{X}$ of the form*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \| & & \downarrow{\scriptstyle g} \\ X & = & X \end{array}$$

in $\mathsf{Alg}_{\mathcal{O}}$*; in other words, suppose* g *is a retraction of* f*. There are natural weak equivalences* $\mathrm{hofib}(f) \simeq \Omega\,\mathrm{hofib}(g)$*; here* Ω *is weakly equivalent, in the underlying category* $\mathsf{Mod}_{\mathcal{R}}$*, to the desuspension* Σ^{-1} *functor.*

Proof. This is because calculating the iterated homotopy fibers of $\mathcal{X}$ in two different ways, by starting in the horizontal direction versus the vertical direction, yields weakly equivalent objects; both calculate the total homotopy fiber of $\mathcal{X}$. In this case, starting in the horizontal direction, followed by the vertical direction, calculates $\mathrm{hofib}(f)$, and starting in the vertical direction, followed by the horizontal direction, calculates $\Omega\,\mathrm{hofib}(g)$, which completes the proof. □

Proposition 6.15 (Proposition 2.9 restated)**.** *Let* Z *be a cosimplicial* $\mathcal{O}$*-algebra and* $n \geq 0$*. There are natural zigzags of weak equivalences*

$$\mathrm{hofib}(\operatorname*{holim}_{\Delta^{\leq n}} Z \to \operatorname*{holim}_{\Delta^{\leq n-1}} Z) \simeq \Omega^n(\text{iterated hofib})\mathcal{Y}_n$$

where $\mathcal{Y}_n$ *denotes the canonical* n*-cube built from the codegeneracy maps of*

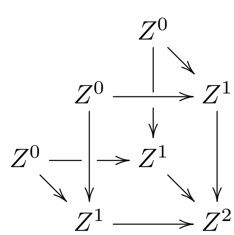
$$Z^0 \xleftarrow{s^0} Z^1 \underset{s^1}{\overset{s^0}{\leftleftarrows}} Z^2 \ \cdots \ Z^n$$

the n*-truncation of* Z*; in particular,* $\mathcal{Y}_0$ *is the object (or* 0*-cube)* Z^0*. Here,* Ω^n *is weakly equivalent, in the underlying category* $\mathsf{Mod}_{\mathcal{R}}$*, to the* n*-fold desuspension* Σ^{-n} *functor. We often refer to* $\mathcal{Y}_n$ *as the* codegeneracy n*-cube associated to* Z*.*

Proof. This follows easily from left cofinality (Propositions 8.27 and 8.30) and the cosimplicial identities, together with repeated application of Proposition 6.14, by using the fact that codegeneracy maps provide retractions for the appropriate coface maps. Alternately, this is proved in [19, X.6.3] for the Tot tower of a Reedy fibrant cosimplicial pointed space, and the same argument verifies it in our context. □

Remark 6.16. For instance, to verify the case $n = 2$, consider the canonical punctured 3-cube built from the coface maps of Z of the form

(39)



where the bottom 2-cube encodes the relation $d^1d^0 = d^0d^0$, the front 2-cube encodes the relation $d^1d^1 = d^2d^1$, and the right-hand 2-cube encodes the relation $d^2d^0 = d^0d^1$. The homotopy limit of this punctured 3-cube is weakly equivalent (Proposition 8.30) to $\operatorname{holim}_{\Delta^{\leq 2}} Z$ and the homotopy limit of the top punctured 2-cube is weakly equivalent to $\operatorname{holim}_{\Delta^{\leq 1}} Z$. By considering iterated homotopy pullbacks, it follows that the homotopy fiber of the induced map $\operatorname{holim}_{\Delta^{\leq 1}} Z \to Z^0$ is weakly equivalent to both $\mathrm{hofib}(\operatorname{holim}_{\Delta^{\leq 2}} Z \to \operatorname{holim}_{\Delta^{\leq 1}} Z)$ and the iterated homotopy fiber of the bottom 2-cube in (39), which itself fits into a commutative diagram of the form

$$
(40) \qquad
\begin{array}{ccccc}
Z^0 & \xrightarrow{d^0} & Z^1 & \xrightarrow{s^0} & Z^0 \\
\downarrow{\scriptstyle d^0} & & \downarrow{\scriptstyle d^0} & & \downarrow{\scriptstyle d^0} \\
Z^1 & \xrightarrow{d^1} & Z^2 & \xrightarrow{s^1} & Z^1 \\
 & & \downarrow{\scriptstyle s^0} & & \downarrow{\scriptstyle s^0} \\
 & & Z^1 & \xrightarrow{s^0} & Z^0
\end{array}
$$

via the cosimplicial identities; note that the bottom-right 2-cube is the codegeneracy 2-cube $\mathcal{Y}_2$ associated to Z. Since the horizontal and vertical composites in (40) are the identity, it follows from Proposition 6.14 that the iterated homotopy fiber of the upper-left 2-cube is weakly equivalent to $\Omega^2(\text{iterated hofib})\mathcal{Y}_2$, and hence we have verified that

$$\operatorname{hofib}(\operatorname*{holim}_{\Delta^{\leq 2}} Z \to \operatorname*{holim}_{\Delta^{\leq 1}} Z) \simeq \Omega^2(\text{iterated hofib})\mathcal{Y}_2$$

For each $n \geq 1$, denote by $\mathcal{Y}_n$ the codegeneracy n-cube associated to $C(Y)$. The reason for introducing $\mathcal{Y}'_n$, $\mathcal{Y}''_n$, $\mathcal{Y}'''_n$, $\mathcal{Y}''''_n$ in Definition 6.17 is that we can homotopically analyze the iterated homotopy fibers of the n-cube $\mathcal{Y}''''_n$ using the combinatorics developed above; since these intermediate n-cubes fit into a zigzag of weak equivalences $\mathcal{Y}_n \simeq \mathcal{Y}''''_n$ of n-cubes, the combinatorial analysis results in a homotopical analysis of $\mathcal{Y}_n$ (Proposition 6.19).

Definition 6.17. If Y is a cofibrant K-coalgebra and $n \geq 1$, denote by $\mathcal{Y}_n$ the codegeneracy n-cube associated to $C(Y)$, and by $\mathcal{Y}'_n$, $\mathcal{Y}''_n$, $\mathcal{Y}'''_n$, $\mathcal{Y}''''_n$ the following weakly equivalent n-cubes. The functor $\mathcal{Y}'_n \colon \mathsf{Cube}_n \to \mathsf{Alg}_J$ is defined by

$$(v_1, v_2, \dots, v_n) \longmapsto J \circ_{A_1} J \circ_{A_2} \cdots J \circ_{A_n} (Y)$$
$$\text{where} \qquad A_i := \begin{cases} \mathcal{O}, & \text{for } v_i = 0, \\ J, & \text{for } v_i = 1, \end{cases}$$

with maps induced by $\mathcal{O} \to J$; the functor $\mathcal{Y}''_n \colon \mathsf{Cube}_n \to \mathsf{Alg}_J$ is defined by

$$(v_1, v_2, \dots, v_n) \longmapsto |\operatorname{Bar}(J, A_1, |\operatorname{Bar}(J, A_2, \dots, |\operatorname{Bar}(J, A_n, Y)| \cdots)|)|$$
$$\text{where} \qquad A_i := \begin{cases} \mathcal{O}, & \text{for } v_i = 0, \\ J, & \text{for } v_i = 1, \end{cases}$$

with maps induced by $\mathcal{O} \to J$; the functor $\mathcal{Y}'''_n \colon \mathsf{Cube}_n \to \mathsf{Alg}_J$ is defined by

$$(v_1, v_2, \dots, v_n) \longmapsto |\operatorname{Bar}(J, A_1, |\operatorname{Bar}(J, A_2, \dots, |\operatorname{Bar}(J, A_n, \tilde{Y})| \cdots)|)|$$
$$\text{where} \qquad A_i := \begin{cases} \mathcal{O}, & \text{for } v_i = 0, \\ J, & \text{for } v_i = 1, \end{cases}$$

with maps induced by $\mathcal{O} \to J$; the functor $\mathcal{Y}''''_n \colon \mathsf{Cube}_n \to \mathsf{Alg}_{\tau_1 \mathcal{O}}$ is defined by

$$(v_1, v_2, \dots, v_n) \longmapsto |\operatorname{Bar}(\tau_1\mathcal{O}, A_1, |\operatorname{Bar}(\tau_1\mathcal{O}, A_2, \dots, |\operatorname{Bar}(\tau_1\mathcal{O}, A_n, \tilde{Y})| \cdots)|)|$$
$$\text{where} \qquad A_i := \begin{cases} \mathcal{O}, & \text{for } v_i = 0, \\ \tau_1\mathcal{O}, & \text{for } v_i = 1, \end{cases}$$

with maps induced by $\mathcal{O} \to \tau_1\mathcal{O}$; here $\tilde{Y} := \tau_1\mathcal{O} \circ_J (Y)$. It is important to note that the natural map $Y \to \tilde{Y}$ in Alg_J is a weak equivalence since $Y \in \mathsf{Alg}_J$ is cofibrant.

Remark 6.18. For instance, $\mathcal{Y}_2$ is the 2-cube associated to $C(Y)$ of the form

$$\begin{array}{ccc} J\circ_{\mathcal{O}} J\circ_{\mathcal{O}}(Y) & \xrightarrow{s^0} & J\circ_{\mathcal{O}}(Y) \\ \downarrow{\scriptstyle s^1} & & \downarrow{\scriptstyle s^0} \\ J\circ_{\mathcal{O}}(Y) & \xrightarrow{s^0} & Y \end{array}$$

$\mathcal{Y}_2'$ is the associated 2-cube of the form

$$\begin{array}{ccc} J\circ_{\mathcal{O}} J\circ_{\mathcal{O}}(Y) & \xrightarrow{s^0} & J\circ_{J} J\circ_{\mathcal{O}}(Y) \\ \downarrow{\scriptstyle s^1} & & \downarrow{\scriptstyle s^0} \\ J\circ_{\mathcal{O}} J\circ_{J}(Y) & \xrightarrow{s^0} & J\circ_{J} J\circ_{J}(Y) \end{array}$$

$\mathcal{Y}_2''$ is the associated 2-cube of the form

$$\begin{array}{ccc} |\operatorname{Bar}(J,\mathcal{O},|\operatorname{Bar}(J,\mathcal{O},Y)|)| & \xrightarrow{s^0} & |\operatorname{Bar}(J,J,|\operatorname{Bar}(J,\mathcal{O},Y)|)| \\ \downarrow{\scriptstyle s^1} & & \downarrow{\scriptstyle s^0} \\ |\operatorname{Bar}(J,\mathcal{O},|\operatorname{Bar}(J,J,Y)|)| & \xrightarrow{s^0} & |\operatorname{Bar}(J,J,|\operatorname{Bar}(J,J,Y)|)| \end{array}$$

$\mathcal{Y}_2'''$ is the associated 2-cube of the form

$$\begin{array}{ccc} |\operatorname{Bar}(J,\mathcal{O},|\operatorname{Bar}(J,\mathcal{O},\tilde{Y})|)| & \xrightarrow{s^0} & |\operatorname{Bar}(J,J,|\operatorname{Bar}(J,\mathcal{O},\tilde{Y})|)| \\ \downarrow{\scriptstyle s^1} & & \downarrow{\scriptstyle s^0} \\ |\operatorname{Bar}(J,\mathcal{O},|\operatorname{Bar}(J,J,\tilde{Y})|)| & \xrightarrow{s^0} & |\operatorname{Bar}(J,J,|\operatorname{Bar}(J,J,\tilde{Y})|)| \end{array}$$

and $\mathcal{Y}_2''''$ the associated 2-cube of the form

$$\begin{array}{ccc} |\operatorname{Bar}(\tau_1\mathcal{O},\mathcal{O},|\operatorname{Bar}(\tau_1\mathcal{O},\mathcal{O},\tilde{Y})|)| & \xrightarrow{s^0} & |\operatorname{Bar}(\tau_1\mathcal{O},\tau_1\mathcal{O},|\operatorname{Bar}(\tau_1\mathcal{O},\mathcal{O},\tilde{Y})|)| \\ \downarrow{\scriptstyle s^1} & & \downarrow{\scriptstyle s^0} \\ |\operatorname{Bar}(\tau_1\mathcal{O},\mathcal{O},|\operatorname{Bar}(\tau_1\mathcal{O},\tau_1\mathcal{O},\tilde{Y})|)| & \xrightarrow{s^0} & |\operatorname{Bar}(\tau_1\mathcal{O},\tau_1\mathcal{O},|\operatorname{Bar}(\tau_1\mathcal{O},\tau_1\mathcal{O},\tilde{Y})|)| \end{array} \tag{41}$$

Proposition 6.19. *If Y is a cofibrant K-coalgebra and $n\geq 1$, then the maps*

$$\mathcal{Y}_n \cong \mathcal{Y}_n' \xleftarrow{\simeq} \mathcal{Y}_n'' \xrightarrow{\simeq} \mathcal{Y}_n''' \xrightarrow{\simeq} \mathcal{Y}_n''''$$

of n-cubes are objectwise weak equivalences. If furthermore, Y is 0-connected, then the iterated homotopy fiber of $\mathcal{Y}_n''''$ is $2n$-connected, and hence the total homotopy fiber of $\mathcal{Y}_n$ is $2n$-connected.

Proof. This is argued exactly as in the proof of Proposition 6.10. By the objectwise calculation of the iterated fibers of the corresponding n-multisimplicial objects, it follows easily but tediously from the combinatorics in Propositions 6.4, 6.5, and 6.6 that the iterated homotopy fiber is weakly equivalent to the realization of a simplicial object which is $(2n-1)$-connected in each simplicial degree, and which is a point in simplicial degree 0; by Proposition 6.3 the realization of such an object is $2n$-connected. To finish off the proof, it suffices to verify that the maps of cubes are objectwise weak equivalences; this follows from Propositions 6.12. □

Remark 6.20. For instance, consider the case $n = 2$. Removing the outer realization in diagram (41), evaluating at (horizontal) simplicial degree r, and calculating the fibers, together with the induced map $(*)$, we obtain the commutative diagram

$$\begin{array}{ccccc} \tau_1\mathcal{O}\circ(\mathcal{O}^{\circ r})^{>1}\circ Z' & \xrightarrow{\subset} & \tau_1\mathcal{O}\circ(\mathcal{O}^{\circ r})\circ Z' & \longrightarrow & \tau_1\mathcal{O}\circ\tau_1(\mathcal{O}^{\circ r})\circ Z' \\ \downarrow{\scriptstyle (*)} & & \downarrow & & \downarrow \\ \tau_1\mathcal{O}\circ(\mathcal{O}^{\circ r})^{>1}\circ Z & \xrightarrow{\subset} & \tau_1\mathcal{O}\circ(\mathcal{O}^{\circ r})\circ Z & \longrightarrow & \tau_1\mathcal{O}\circ\tau_1(\mathcal{O}^{\circ r})\circ Z \end{array} \tag{42}$$

where $Z' := |\operatorname{Bar}(\tau_1\mathcal{O},\mathcal{O},\tilde{Y})|$, $Z := |\operatorname{Bar}(\tau_1\mathcal{O},\tau_1\mathcal{O},\tilde{Y})|$, $\tau_1(\mathcal{O}^{\circ r})\cong(\tau_1\mathcal{O})^{\circ r}$, and the rows are cofiber sequences in $\mathsf{Mod}_{\mathcal{R}}$. We want to calculate the fiber of the map $(*)$. Removing the realization from $(*)$ in diagram (42) and evaluating at (vertical) simplicial degree s, we obtain the commutative diagram

$$\begin{array}{ccccc} (\text{iterated hofib})_{r,s} = \tau_1\mathcal{O}\circ\big(\widetilde{W(r,s)}\big)^{>2}\circ\tilde{Y} & \xrightarrow{\subset} & \tau_1\mathcal{O}\circ W(r,s)\circ\mathcal{O}^{\circ s}\circ\tilde{Y} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \tau_1\mathcal{O}\circ W(r,s)\circ\tau_1(\mathcal{O}^{\circ s})\circ\tilde{Y} \end{array}$$

where

$$\begin{aligned} W(r,s) &:= (\mathcal{O}^{\circ r})^{>1}\circ\tau_1\mathcal{O} \\ \widetilde{W(r,s)} &:= \coprod_{t>1} W(r)[t]\wedge_{\Sigma_t}((\mathcal{O}^{\circ s})^{\check{\otimes} t})^{>t} \end{aligned}$$

The indicated square is a pushout diagram in $\mathsf{Mod}_{\mathcal{R}}$ and a pullback diagram in $\mathsf{Alg}_{\mathcal{O}}$; here, we used Propositions 6.5 and 6.6. Since $\tilde{Y}$ is 0-connected, it follows that $(\text{iterated hofib})_{r,s}$ is 2-connected, $(\text{iterated hofib})_{0,s} = *$, and $(\text{iterated hofib})_{r,0} = *$, for each $r, s \geq 0$. In other words, the iterated homotopy fiber, after removing the horizontal and vertical realizations, is a bisimplicial $\mathcal{O}$-algebra of the form (not showing the face or degeneracy maps)

$$\begin{array}{cccc} \vdots & & & \\ * & \square & \square & \\ * & \square & \square & \\ * & * & * & \cdots \end{array}$$

where each box $\square$ indicates a 2-connected $\mathcal{O}$-algebra. Applying realization in the vertical direction gives a (horizontal) simplicial $\mathcal{O}$-algebra that is 3-connected in each simplicial degree r, and is the null object $*$ in simplicial degree 0, and therefore realization in the horizontal direction gives an $\mathcal{O}$-algebra that is 4-connected; hence the total homotopy fiber of $\mathcal{Y}_2''''$ is 4-connected.

Proposition 6.21 (Proposition 2.11 restated)**.** *Let Y be a cofibrant K-coalgebra and $n \geq 1$. Denote by $\mathcal{Y}_n$ the codegeneracy n-cube associated to the cosimplicial cobar construction $C(Y)$ of Y. If Y is 0-connected, then the total homotopy fiber of $\mathcal{Y}_n$ is $2n$-connected.*

Proof. This is a special case of Proposition 6.19. □

6.22. **Commuting TQ-homology past homotopy limits over Δ.** The purpose of this section is to prove Theorem 2.14, which provides connectivity estimates for the comparison map $\mathsf{L}Q\,\mathrm{holim}_{\Delta^{\leq n}} C(Y) \rightarrow \mathrm{holim}_{\Delta^{\leq n}} \mathsf{L}Q\, C(Y)$.

The following definitions and constructions appear in [53] in the context of spaces, and will also be useful in our context when working with the spectral algebra higher Blakers-Massey theorems proved in [25].

Definition 6.23 (Indexing categories for cubical diagrams)**.** Let W be a finite set and M a category.

- Denote by $\mathcal{P}(W)$ the poset of all subsets of W, ordered by inclusion $\subset$ of sets. We will often regard $\mathcal{P}(W)$ as the category associated to this partial order in the usual way; the objects are the elements of $\mathcal{P}(W)$, and there is a morphism $U{\rightarrow}V$ if and only if $U \subset V$.
- Denote by $\mathcal{P}_0(W) \subset \mathcal{P}(W)$ the poset of all nonempty subsets of W; it is the full subcategory of $\mathcal{P}(W)$ containing all objects except the initial object $\emptyset$.
- A W-*cube* $\mathcal{X}$ in M is a $\mathcal{P}(W)$-shaped diagram $\mathcal{X}$ in M; in other words, a functor $\mathcal{X}\colon\ \mathcal{P}(W){\rightarrow}\mathsf{M}$.

Remark 6.24. If $n = |W|$ and $\mathcal{X}$ is a W-cube in M, we will sometimes refer to $\mathcal{X}$ simply as an n-*cube* in M. In particular, a 0-cube is an object in M and a 1-cube is a morphism in M.

Definition 6.25 (Faces of cubical diagrams)**.** Let W be a finite set and M a category. Let $\mathcal{X}$ be a W-cube in M and consider any subsets $U \subset V \subset W$. Denote by $\partial_U^V \mathcal{X}$ the $(V-U)$-cube defined objectwise by

$$T \mapsto (\partial_U^V \mathcal{X})_T := \mathcal{X}_{T\cup U}, \qquad T \subset V-U.$$

In other words, $\partial_U^V \mathcal{X}$ is the $(V-U)$-cube formed by all maps in $\mathcal{X}$ between $\mathcal{X}_U$ and $\mathcal{X}_V$. We say that $\partial_U^V \mathcal{X}$ is a *face* of $\mathcal{X}$ of *dimension* $|V-U|$.

Definition 6.26. Let $Z \in (\mathsf{Alg}_{\mathcal{O}})^{\Delta}$ and $n \geq 0$. Assume that Z is objectwise fibrant and denote by $Z\colon\ \mathcal{P}_0([n]){\rightarrow}\mathsf{Alg}_{\mathcal{O}}$ the composite

$$\mathcal{P}_0([n]) \rightarrow \Delta^{\leq n} \rightarrow \Delta \rightarrow \mathsf{Alg}_{\mathcal{O}}$$

or "restriction" to $\mathcal{P}_0([n])$. The *associated* ∞-*cartesian* $(n+1)$-*cube built from* Z, denoted $\widetilde{Z}\colon\ \mathcal{P}([n]){\rightarrow}\mathsf{Alg}_{\mathcal{O}}$, is defined objectwise by

$$\widetilde{Z}_V := \begin{cases} \mathrm{holim}^{\mathsf{BK}}_{T\neq\emptyset} Z_T, & \text{for } V=\emptyset,\\ Z_V, & \text{for } V\neq\emptyset. \end{cases}$$

It is important to note (Proposition 8.30) that there are natural weak equivalences

$$\underset{\Delta^{\leq n}}{\mathrm{holim}}\, Z \simeq \mathrm{holim}^{\mathsf{BK}}_{T\neq\emptyset} Z_T = \widetilde{Z}_\emptyset$$

in $\mathsf{Alg}_{\mathcal{O}}$.

The following is closely related to Munson-Volic [89, 5.5.7] and Sinha [106, 7.2] and the same argument verifies it in this context.

Proposition 6.27 (Comparison with the codegeneracy cubes)**.** *Let $Z \in (\mathsf{Alg}_{\mathcal{O}})^{\Delta}$ and $n \geq 0$. Assume that Z is objectwise fibrant. Let $\emptyset \neq T \subset [n]$ and $t \in T$. Then there is a weak equivalence*

$$\text{(iterated hofib)}\partial^T_{\{t\}}\widetilde{Z} \simeq \Omega^{|T|-1}\text{(iterated hofib)}\mathcal{Y}_{|T|-1}$$

in $\mathsf{Alg}_{\mathcal{O}}$, where $\mathcal{Y}_{|T|-1}$ denotes the codegeneracy $(|T|-1)$-cube associated to Z. Here, Ω^k is weakly equivalent, in the underlying category $\mathsf{Mod}_{\mathcal{R}}$, to the k-fold desuspension Σ^{-k} functor for each $k \geq 0$.

Proof. It follows easily from the cosimplicial identities that $\partial^T_{\{t\}}\widetilde{Z}$ is connected to $\mathcal{Y}_{|T|-1}$ by a sequence of retractions, built from codegeneracy maps, that have the following special property: the retractions fit into a commutative diagram made up of a sequence of concatenated $(|T|-1)$-cubes, starting with $\partial^T_{\{t\}}\widetilde{Z}$, such that each added cube is in a distinct direction (the concatenation direction of the added cube), and such that each arrow in the added cube, whose direction is in the concatenation direction, is a retraction of the preceding arrow in that direction (in the previous cube in the sequence), with the last $(|T|-1)$-cube in the sequence composed entirely of codegeneracy maps; this is elaborated in Remark 6.29. The resulting commutative diagram is a sequence of $|T|$ concatenated $(|T|-1)$-cubes that starts with $\partial^T_{\{t\}}\widetilde{Z}$ and ends with $\mathcal{Y}_{|T|-1}$. Repeated application of Proposition 6.14 finishes the proof. ☐

Consider the following special case.

Remark 6.28. For instance, suppose $n = 2$ and $T = \{0,1,2\}$. In the case $t = 0$, consider the following left-hand commutative diagram

(43)
$$\begin{array}{ccccc} Z_{\{0\}} & \xrightarrow{d^1} & Z_{\{0,1\}} & \xrightarrow{s^0} & Z_{\{0\}} \\ \downarrow{\scriptstyle d^1} & (*)' & \downarrow{\scriptstyle d^2} & & \downarrow{\scriptstyle d^1} \\ Z_{\{0,2\}} & \xrightarrow{d^1} & Z_{\{0,1,2\}} & \xrightarrow{s^0} & Z_{\{0,2\}} \\ & & \downarrow{\scriptstyle s^1} & & \downarrow{\scriptstyle s^0} \\ & & Z_{\{0,1\}} & \xrightarrow{s^0} & Z_{\{0\}} \end{array} \qquad \begin{array}{ccccc} Z_{\{1\}} & \xrightarrow{d^0} & Z_{\{0,1\}} & \xrightarrow{s^0} & Z_{\{1\}} \\ \downarrow{\scriptstyle d^1} & (*)'' & \downarrow{\scriptstyle d^2} & & \downarrow{\scriptstyle d^1} \\ Z_{\{1,2\}} & \xrightarrow{d^0} & Z_{\{0,1,2\}} & \xrightarrow{s^0} & Z_{\{1,2\}} \\ \downarrow{\scriptstyle s^0} & & \downarrow{\scriptstyle s^1} & & \downarrow{\scriptstyle s^0} \\ Z_{\{1\}} & \xrightarrow{d^0} & Z_{\{0,1\}} & \xrightarrow{s^0} & Z_{\{1\}} \end{array}$$

in the case $t = 1$, consider the right-hand commutative diagram in (43), and in the case $t = 2$, consider the commutative diagram

$$\begin{array}{ccccc} Z_{\{2\}} & \xrightarrow{d^0} & Z_{\{0,2\}} & & \\ \downarrow{\scriptstyle d^0} & (*)''' & \downarrow{\scriptstyle d^1} & & \\ Z_{\{1,2\}} & \xrightarrow{d^0} & Z_{\{0,1,2\}} & \xrightarrow{s^0} & Z_{\{1,2\}} \\ \downarrow{\scriptstyle s^0} & & \downarrow{\scriptstyle s^1} & & \downarrow{\scriptstyle s^0} \\ Z_{\{2\}} & \xrightarrow{d^0} & Z_{\{0,2\}} & \xrightarrow{s^0} & Z_{\{2\}} \end{array}$$

The upper left-hand squares $(*)'$, $(*)''$, $(*)'''$ are the 2-cubes $\partial^T_{\{0\}}\widetilde{Z}$, $\partial^T_{\{1\}}\widetilde{Z}$, $\partial^T_{\{2\}}\widetilde{Z}$, respectively, and the lower right-hand squares are each a copy of the codegeneracy 2-cube $\mathcal{Y}_2$. By repeated application of Proposition 6.14, it follows easily that there are weak equivalences

$$(\text{iterated hofib})\partial^T_{\{t\}}\widetilde{Z} \simeq \Omega^2(\text{iterated hofib})\mathcal{Y}_2$$

in $\mathsf{Alg}_{\mathcal{O}}$ for each $t \in T$. Note that there are two distinct paths connecting $\partial^T_{\{1\}}\widetilde{Z}$ with $\mathcal{Y}_2$, but there is only one path connecting $\partial^T_{\{0\}}\widetilde{Z}$ with $\mathcal{Y}_2$; similarly, there is only one path connecting $\partial^T_{\{2\}}\widetilde{Z}$ with $\mathcal{Y}_2$. The proof of Proposition 6.27 is simply the observation, which follows easily from the cosimplicial identities, that one such path always exists.

Remark 6.29. The construction in the proof of Proposition 6.27 can be elaborated as follows. Let $n \geq 1$. Suppose that $\emptyset \neq T \subset [n]$ and $t \in T$. For each $\emptyset \neq S \subset T$, denote by $\mathcal{P}^T_S$ the poset of all subsets of T that contain S, ordered by inclusion $\subset$ of sets; it is the full subcategory of $\mathcal{P}_0(T) \subset \mathcal{P}_0([n])$ containing all objects that contain the set S. Define $\widetilde{Z}^T_S$ to be the restriction of $\widetilde{Z}$ to $\mathcal{P}^T_S$; it is important to note that $\widetilde{Z}^T_S$ is a copy of the cube $\partial^T_S\widetilde{Z}$.

Denote by Δ_{large} the category of nonempty totally-ordered finite sets and order-preserving maps; note that $\Delta \subset \Delta_{\text{large}}$ is a skeletal subcategory and there is an equivalence of categories $\Delta_{\text{large}} {\rightarrow} \Delta$. Consider the functor $\sigma\colon \big(\mathcal{P}^T_{\{t\}}\big)^{\text{op}} {\rightarrow} \Delta_{\text{large}}$ which on objects is defined by $\sigma(V) := V$ and on arrows is defined as follows. It maps each $V \subset W$ in $\mathcal{P}^T_{\{t\}}$ to the order-preserving function $\sigma_{W,V}\colon W {\rightarrow} V$ defined by

$$\sigma_{W,V}(w) := \begin{cases} \max\{v \in V : v \leq w\}, & \text{for } t \leq w, \\ \min\{v \in V : v \geq w\}, & \text{for } t \geq w. \end{cases}$$

Note that $\sigma_{W,V}(v) = v$ for each $v \in V$. Define $\mathcal{Y}^T_{\{t\}}$ to be the composite

$$\big(\mathcal{P}^T_{\{t\}}\big)^{\text{op}} \xrightarrow{\sigma} \Delta_{\text{large}} {\rightarrow} \Delta \xrightarrow{Z} \mathsf{Alg}_{\mathcal{O}}$$

Then $\mathcal{Y}^T_{\{t\}}$ is a $(|T|-1)$-cube with the same vertices as $\widetilde{Z}^T_{\{t\}}$, but with edges going in the opposite direction; it is important to note that $\mathcal{Y}^T_{\{t\}}$ is a copy of the codegeneracy cube $\mathcal{Y}_{|T|-1}$.

By construction, for each $V \subset W$ in $\mathcal{P}^T_{\{t\}}$ the composite

$$\big(\widetilde{Z}^T_{\{t\}}\big)_V \longrightarrow \big(\widetilde{Z}^T_{\{t\}}\big)_W = \big(\mathcal{Y}^T_{\{t\}}\big)_W \longrightarrow \big(\mathcal{Y}^T_{\{t\}}\big)_V = \big(\widetilde{Z}^T_{\{t\}}\big)_V$$

is the identity map; in other words, we have built a collection of retracts from the coordinate free description of the codegeneracy maps. It follows from this construction that $\widetilde{Z}^T_{\{t\}}$ is connected to $\mathcal{Y}^T_{\{t\}}$ by a sequence of retractions, built from codegeneracy maps. The resulting commutative diagram is a sequence of $|T|$ concatenated $(|T|-1)$-cubes that starts with $\widetilde{Z}^T_{\{t\}}$ and ends with $\mathcal{Y}^T_{\{t\}}$.

Proposition 6.30. *Let $n \geq 0$. Suppose that $\emptyset \neq T \subset [n]$ and $t \in T$. If $Y \in \mathsf{coAlg}_{\mathsf{K}}$ is cofibrant and 0-connected, then the cube*

$$\partial^T_{\{t\}}\widetilde{\mathfrak{C}(Y)} \quad \textit{is } |T|\textit{-cartesian.}$$

Proof. We know by Proposition 6.19 that the iterated homotopy fiber of $\mathcal{Y}_{|T|-1}$ is $2(|T|-1)$-connected. Hence by Proposition 6.27 we know that the iterated homotopy fiber of $\partial^T_{\{t\}}\widetilde{\mathfrak{C}(Y)}$ is $(|T|-1)$-connected; it follows that $\partial^T_{\{t\}}\widetilde{\mathfrak{C}(Y)}$ is $|T|$-cartesian. □

Proposition 6.31. *Let $n \geq 0$. Suppose that $\emptyset \neq T \subset [n]$ and $\emptyset \neq S \subset T$. If $Y \in \mathsf{coAlg}_{\mathsf{K}}$ is cofibrant and 0-connected, then the cube*

$$\partial_S^T \widetilde{\mathfrak{C}(Y)} \quad \text{is } (|T|-|S|+1)\text{-cartesian.} \tag{44}$$

Proof. We want to verify (44) for each $\emptyset \neq S \subset T$. We know that (44) is true for $|S| = 1$ by Proposition 6.30. We will argue by upward induction on $|S|$. Let $t \in S$ and note that the cube $\partial_{S-\{t\}}^T \widetilde{\mathfrak{C}(Y)}$ can be written as the composition of cubes

$$\partial_{S-\{t\}}^{T-\{t\}} \widetilde{\mathfrak{C}(Y)} \longrightarrow \partial_S^T \widetilde{\mathfrak{C}(Y)}$$

We know by the induction hypothesis that the composition is $(|T|-|S|+2)$-cartesian and that the left-hand cube is $(|T|-|S|+1)$-cartesian, hence it follows [25, 3.8, 3.10] that the right-hand cube is $(|T|-|S|+1)$-cartesian, which finishes the argument that (44) is true for each $\emptyset \neq S \subset T$. □

Theorem 6.32. *Let $Y \in \mathsf{coAlg}_{\mathsf{K}}$ be cofibrant and $n \geq 1$. Consider the ∞-cartesian $(n+1)$-cube $\widetilde{\mathfrak{C}(Y)}$ in $\mathsf{Alg}_{\mathcal{O}}$ built from $\mathfrak{C}(Y)$. If Y is 0-connected, then*

(a) *the cube $\widetilde{\mathfrak{C}(Y)}$ is $(2n+4)$-cocartesian in $\mathsf{Alg}_{\mathcal{O}}$,*
(b) *the cube $FQc\widetilde{\mathfrak{C}(Y)}$ is $(2n+4)$-cocartesian in Alg_J,*
(c) *the cube $FQc\widetilde{\mathfrak{C}(Y)}$ is $(n+4)$-cartesian in Alg_J.*

Proof. Consider part (a) and let $W = [n]$. We want to use the higher dual Blakers-Massey theorem for structured ring spectra [25, 1.11] to estimate how close the W-cube $\widetilde{\mathfrak{C}(Y)}$ in $\mathsf{Alg}_{\mathcal{O}}$ is to being cocartesian. We know from Proposition 6.31 that for each nonempty subset $V \subset W$, the V-cube $\partial_{W-V}^W \widetilde{\mathfrak{C}(Y)}$ is $(|V|+1)$-cartesian; it is ∞-cartesian by construction when $V = W$. Hence it follows from [25, 1.11] that $\widetilde{\mathfrak{C}(Y)}$ is $(2(n+1)+2)$-cocartesian in $\mathsf{Alg}_{\mathcal{O}}$, which finishes the proof of part (a). Part (b) follows from the fact that $Q\colon \mathsf{Alg}_{\mathcal{O}} \rightarrow \mathsf{Alg}_J$ is a left Quillen functor together with the relative TQ-Hurewicz theorem in [58, 1.9]. Part (c) follows from the fact that Alg_J and $\mathsf{Alg}_{\tau_1 \mathcal{O}} \cong \mathsf{Mod}_{\mathcal{O}[1]}$ are Quillen equivalent (Section 3) via the change of operads adjunction along $J \rightarrow \tau_1 \mathcal{O}$, together with [25, 3.10]. □

Proof of Theorem 2.14. We want to verify that the comparison map

$$\mathsf{L}Q \operatorname{holim}_{\Delta^{\leq n}} C(Y) \longrightarrow \operatorname{holim}_{\Delta^{\leq n}} \mathsf{L}Q\, C(Y),$$

is $(n+4)$-connected, which is equivalent to verifying that $\mathsf{L}Q\widetilde{\mathfrak{C}(Y)}$ is $(n+4)$-cartesian. Since $\mathsf{L}Q \simeq FQc$, Theorem 6.32(c) completes the proof. □

Remark 6.33. For the convenience of the reader, we will further elaborate on the connection between the maps appearing in the proof of Theorem 1.2 and the derived counit map. Consider the following commutative diagram

$$\begin{array}{ccccc}
\mathsf{L}Q\,\mathrm{holim}_{\Delta}\, C(Y) & \xrightarrow{(*)'} & \mathrm{holim}_{\Delta}\, \mathsf{L}Q\, C(Y) & \xrightarrow{(*)''} & Y \\
\downarrow{\simeq} & & \downarrow{\simeq} & & \\
FQc\,\mathrm{holim}^{\mathsf{BK}}_{\Delta}\, \mathfrak{C}(Y) & \longrightarrow & \mathrm{holim}^{\mathsf{BK}}_{\Delta}\, FQ\mathfrak{C}(Y) & \searrow & (**)\downarrow{\simeq} \\
(*)\downarrow{\simeq} & & (*)\downarrow{\simeq} & & \\
FQc\,\mathrm{holim}^{\mathsf{BK}}_{\Delta_{\mathsf{res}}}\, \mathfrak{C}(Y) & \longrightarrow & \mathrm{holim}^{\mathsf{BK}}_{\Delta_{\mathsf{res}}}\, FQ\mathfrak{C}(Y) & \searrow & \\
(*)\uparrow{\simeq} & & (*)\uparrow{\simeq} & & \\
FQc\,\mathrm{Tot}^{\mathsf{res}}\, \mathfrak{C}(Y) & \longrightarrow & \mathrm{Tot}^{\mathsf{res}}\, FQ\mathfrak{C}(Y) & \longrightarrow & FY \\
(**)\uparrow{\simeq} & & & \nearrow{(\#)} & \\
Qc\,\mathrm{Tot}^{\mathsf{res}}\, \mathfrak{C}(Y) & & & &
\end{array}$$

where the arrow (#) is defined by the indicated composition; here, the top horizontal maps are the maps in (11). The map underlying the derived counit map is precisely the map (#). The maps $(*)$ are weak equivalences by Proposition 8.17, the maps $(**)$ are weak equivalences since each is the unit of the simplicial fibrant replacement monad F. Hence to verify that the map (#) underlying the derived counit map is a weak equivalence, it suffices to verify that $(*)'$ and $(*)''$ are weak equivalences; this is verified in the proof of Theorem 1.2. This reduction argument can be thought of as a homotopical Barr-Beck comonadicity theorem; see [3, 2.20].

7. Appendix: Simplicial structures on $\mathcal{O}$-algebras

The simplicial structure on $\mathcal{O}$-algebras is established in Elmendorf-Kriz-Mandell-May [37, VII.2.10]; for a development in the context of symmetric spectra, see [58, 6.1].

Definition 7.1. Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules. Let X, X' be $\mathcal{O}$-algebras and K a simplicial set. The *mapping object* $\mathbf{hom}_{\mathsf{Alg}_{\mathcal{O}}}(K, X)$ in $\mathsf{Alg}_{\mathcal{O}}$ is defined by

$$\mathbf{hom}_{\mathsf{Alg}_{\mathcal{O}}}(K, X) := \mathrm{Map}(K_+, X)$$

with left $\mathcal{O}$-action map induced by $m\colon \mathcal{O}\circ(X)\to X$, together with the natural maps $K \to K^{\times t}$ in sSet for $t \geq 0$; these are the diagonal maps for $t \geq 1$ and the constant map for $t = 0$. For ease of notation purposes, we sometimes drop the $\mathsf{Alg}_{\mathcal{O}}$ decoration from the notation and simply denote the mapping object by $\mathbf{hom}(K, X)$.

Definition 7.2. Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules. Let X, X' be $\mathcal{O}$-algebras and K a simplicial set. The *tensor product* $X\dot{\otimes}K$ in $\mathsf{Alg}_{\mathcal{O}}$ is defined by the reflexive coequalizer

$$X\dot{\otimes}K := \mathrm{colim}\Big(\ \mathcal{O}\circ(X\wedge K_+) \underset{d_1}{\overset{d_0}{\leftleftarrows}} \mathcal{O}\circ\big(\mathcal{O}\circ(X)\wedge K_+\big)\ \Big) \tag{45}$$

in $\mathsf{Alg}_{\mathcal{O}}$, with d_0 induced by operad multiplication $m\colon \mathcal{O}\circ\mathcal{O}\to\mathcal{O}$ and the natural map $\nu\colon \mathcal{O}\circ(X)\wedge K_+ \to \mathcal{O}\circ(X\wedge K_+)$, while d_1 is induced by the left $\mathcal{O}$-action map $m\colon \mathcal{O}\circ(X)\to X$.

Definition 7.3. Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules. Let X, X' be $\mathcal{O}$-algebras. The *mapping space* $\mathbf{Hom}_{\mathsf{Alg}_{\mathcal{O}}}(X,X')$ in sSet is defined degreewise by

$$\mathbf{Hom}_{\mathsf{Alg}_{\mathcal{O}}}(X,X')_n := \hom_{\mathsf{Alg}_{\mathcal{O}}}(X\dot{\otimes}\Delta[n],X')$$

For ease of notation purposes, we often drop the $\mathsf{Alg}_{\mathcal{O}}$ decoration from the notation and simply denote the mapping space by $\mathbf{Hom}(X,X')$.

Proposition 7.4. *Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules. Consider $\mathsf{Alg}_{\mathcal{O}}$ with the positive flat stable model structure. Then $\mathsf{Alg}_{\mathcal{O}}$ is a simplicial model category (see* [51, II.3]*) with the above definitions of mapping object, tensor product, and mapping space.*

Remark 7.5. In particular, there are isomorphisms

$$\begin{aligned} \hom_{\mathsf{Alg}_{\mathcal{O}}}(X\dot{\otimes}K,X') &\cong \hom_{\mathsf{Alg}_{\mathcal{O}}}(X,\mathbf{hom}(K,X')) \\ &\cong \hom_{\mathsf{sSet}}(K,\mathbf{Hom}(X,X')) \end{aligned} \tag{46}$$

in Set, natural in X,K,X', that extend to isomorphisms

$$\begin{aligned} \mathbf{Hom}_{\mathsf{Alg}_{\mathcal{O}}}(X\dot{\otimes}K,X') &\cong \mathbf{Hom}_{\mathsf{Alg}_{\mathcal{O}}}(X,\mathbf{hom}(K,X')) \\ &\cong \mathbf{Hom}_{\mathsf{sSet}}(K,\mathbf{Hom}(X,X')) \end{aligned}$$

in sSet, natural in X,K,X'.

The following proposition verifies that the change of operads adjunction (12) meshes nicely with the simplicial structure; this is closely related to [51, II.2.9].

Proposition 7.6. *Let $f\colon \mathcal{O}\to\mathcal{O}'$ be a map of operads in $\mathcal{R}$-modules. Let X be an $\mathcal{O}$-algebra, Y an $\mathcal{O}'$-algebra, and K,L simplicial sets. Then*

(a) *there is a natural isomorphism $\sigma\colon f_*(X)\dot{\otimes}K \xrightarrow{\cong} f_*(X\dot{\otimes}K)$;*
(b) *there is an isomorphism*

$$\mathbf{Hom}(f_*(X),Y) \cong \mathbf{Hom}(X,f^*(Y))$$

in sSet, natural in X,Y, that extends the adjunction isomorphism in (12)*;*
(c) *there is an isomorphism*

$$f^*\,\mathbf{hom}(K,Y) = \mathbf{hom}(K,f^*Y)$$

in $\mathsf{Alg}_{\mathcal{O}}$, natural in K,Y;
(d) *there is a natural map $\sigma\colon f^*(Y)\dot{\otimes}K\to f^*(Y\dot{\otimes}K)$ induced by f;*
(e) *the functors f_* and f^* are simplicial functors (Remark 7.7) with the structure maps σ of (a) and (d), respectively.*

Remark 7.7. For a useful reference on simplicial functors in the context of homotopy theory, see [61, 9.8.5].

Proof. Part (c) follows from the observation that the underlying $\mathcal{R}$-modules are identical and the $\mathcal{O}$-action maps are the same. Consider part (a). This follows easily from part (c), together with the Yoneda lemma, by verifying there are natural isomorphisms

$$\hom(f_*(X\dot{\otimes}K),Y) \xrightarrow[\cong]{\varphi} \hom(f_*(X)\dot{\otimes}K,Y).$$

In particular, σ is the image under φ of the identity map on $f_*(X\dot{\otimes}K)$ and σ^{-1} is the image under φ^{-1} of the identity map on $f_*(X)\dot{\otimes}K$. Consider part (b). The indicated isomorphism of simplicial sets is defined objectwise by the composition of natural isomorphisms

$$\hom(f_*(X)\dot{\otimes}\Delta[n],Y)\cong \hom(f_*(X\dot{\otimes}\Delta[n]),Y)\cong \hom(X\dot{\otimes}\Delta[n],f^*(Y)),$$

where the left-hand isomorphism is the map $(\sigma^{-1},\mathrm{id})$. Consider part (d). The map σ is the map induced by $f\circ\mathrm{id}\colon \mathcal{O}\circ(Y\wedge K_+)\to\mathcal{O}'\circ(Y\wedge K_+)$ via the colimit description of tensor product in (45); it will be useful to note that σ can also be described as the image of the identity map on $Y\dot{\otimes}K$ under the composition of maps

$$\begin{aligned}\hom(Y\dot{\otimes}K,Y')\cong \hom(Y,\mathbf{hom}(K,Y'))&\xrightarrow{f^*}\hom\big(f^*(Y),\mathbf{hom}(K,f^*(Y'))\big)\\ &\cong \hom(f^*(Y)\dot{\otimes}K,f^*(Y'))\end{aligned}$$

Here, we used the identification in part (c). Consider part (e). It suffices to verify that corresponding diagrams (1) and (2) in [61, 9.8.5] commute. In the case of f_*, this follows from the fact that the structure maps are the canonical isomorphisms. Consider the case of f^*. Verifying that diagram (1) in [61, 9.8.5] commutes follows most easily from using the colimit description in (45), and verifying that the following diagram

$$\begin{array}{ccccc} f^*\big(Y\dot{\otimes}(K\times L)\big) & \xleftarrow{\sigma} & f^*(Y)\dot{\otimes}(K\times L) & \xleftarrow{\cong} & \big(f^*(Y)\dot{\otimes}K\big)\dot{\otimes}L \\ {\scriptstyle\cong}\big\uparrow & & & & \big\downarrow{\scriptstyle\sigma\dot{\otimes}\mathrm{id}} \\ f^*\big((Y\dot{\otimes}K)\dot{\otimes}L\big) & & \xleftarrow{\qquad\sigma\qquad} & & f^*(Y\dot{\otimes}K)\dot{\otimes}L \end{array} \tag{47}$$

commutes—this corresponds to diagram (2) in [61, 9.8.5]—follows similarly from (45), together with the fact that f respects the operad multiplication maps $\mathcal{O}\circ\mathcal{O}\to\mathcal{O}$ and $\mathcal{O}'\circ\mathcal{O}'\to\mathcal{O}'$. □

The following proposition verifies that the natural transformations (see (14)) associated to the TQ-homology spectrum functor respect the simplicial structure maps.

Proposition 7.8. *Let $f\colon\mathcal{O}\to\mathcal{O}'$ be a map of operads in $\mathcal{R}$-modules. Consider the monad f^*f_* on $\mathsf{Alg}_{\mathcal{O}}$ and the comonad f_*f^* on $\mathsf{Alg}_{\mathcal{O}'}$ associated to the adjunction (f_*,f^*) in (12). The associated natural transformations*

$$\mathrm{id}\xrightarrow{\eta} f^*f_*\quad(\textit{unit}),\qquad\qquad \mathrm{id}\xleftarrow{\varepsilon} f_*f^*\quad(\textit{counit}),$$

$$f^*f_*f^*f_*\to f^*f_*\quad(\textit{multiplication}),\qquad\qquad f_*f^*f_*f^*\xleftarrow{m} f_*f^*\quad(\textit{comultiplication})$$

are simplicial natural transformations.

Proof. Consider the case of the unit map. It suffices to verify that the diagram

$$\begin{array}{ccccc} X\dot{\otimes}K & & \overline{\overline{\qquad\qquad\qquad}} & & X\dot{\otimes}K \\ \big\downarrow{\scriptstyle\eta_X\dot{\otimes}\mathrm{id}} & & & & \big\downarrow{\scriptstyle\eta_{X\dot{\otimes}K}} \\ f^*f_*(X)\dot{\otimes}K & \xrightarrow[\sigma]{} & f^*(f_*(X)\dot{\otimes}K) & \xleftarrow[f^*(\sigma^{-1})]{\cong} & f^*f_*(X\dot{\otimes}K) \end{array}$$

commutes; hence it suffices to verify that the two composite maps of the form $X\dot{\otimes}K\rightrightarrows f^*(f_*(X)\dot{\otimes}K)$ are identical. This follows by working with the hom-set description of the indicated σ^{-1} and σ maps (see the proof of Proposition 7.6) and noting that each arrow is the image of the identity map on $f_*(X)\dot{\otimes}K$ under the composition of natural isomorphisms

$$\begin{aligned}\hom(f_*(X)\dot{\otimes}K,Y) &\cong \hom(f_*(X),\mathbf{hom}(K,Y)) \cong \hom(X,f^*\,\mathbf{hom}(K,Y))\\ &= \hom(X,\mathbf{hom}(K,f^*Y)) \cong \hom(X\dot{\otimes}K,f^*Y).\end{aligned}$$

Consider the case of the counit map. It suffices to verify that the diagram

$$\begin{array}{ccccc} f_*f^*(Y)\dot{\otimes}K & \underset{\cong}{\xleftarrow{\sigma^{-1}}} & f_*(f^*(Y)\dot{\otimes}K) & \xrightarrow{f_*(\sigma)} & f_*f^*(Y\dot{\otimes}K)\\ \downarrow{\scriptstyle \varepsilon_Y\dot{\otimes}\mathrm{id}} & & & & \downarrow{\scriptstyle \varepsilon_{Y\dot{\otimes}K}}\\ Y\dot{\otimes}K & = & = & = & Y\dot{\otimes}K \end{array}$$

commutes; this follows, similar to above, by noting that the two composite maps of the form $f_*(f^*(Y)\dot{\otimes}K)\rightrightarrows Y\dot{\otimes}K$ are each equal to the image of the identity map on $Y\dot{\otimes}K$ under the composition of natural maps

$$\begin{aligned}\hom(Y\dot{\otimes}K,Y') &\cong \hom(Y,\mathbf{hom}(K,Y')) \xrightarrow{f^*} \hom\big(f^*(Y),\mathbf{hom}(K,f^*(Y'))\big)\\ &\cong \hom(f^*(Y)\dot{\otimes}K,f^*(Y')) \cong \hom(f_*(f^*(Y)\dot{\otimes}K),Y')\end{aligned}$$

Here we have used the identification in Proposition 7.6(c). The remaining two cases follow from the fact that the multiplication map is $f^*\varepsilon f_*$, the comultiplication map is $f_*\eta f^*$, and composing a simplicial natural transformation with a simplicial functor, on the left or right, gives a simplicial natural transformation. □

8. Appendix: Homotopy limit towers and cosimplicial $\mathcal{O}$-algebras

The purpose of this appendix is to make precise the several towers of $\mathcal{O}$-algebras, associated to a given cosimplicial $\mathcal{O}$-algebra, that are needed in this paper. Most of the arguments involve holim_Δ and $\mathrm{holim}_{\Delta^{\leq n}}$, which are defined in terms of the Bousfield-Kan homotopy limit functors $\mathrm{holim}^{\mathsf{BK}}_\Delta$ and $\mathrm{holim}^{\mathsf{BK}}_{\Delta^{\leq n}}$, which in turn are defined in terms of the totalization functor Tot for cosimplicial $\mathcal{O}$-algebras. For technical reasons that arise in the homotopy theory of K-coalgebras, we also require use of the restricted totalization $\mathrm{Tot}^{\mathsf{res}}$ functor. Furthermore, the construction of the homotopy spectral sequence associated to a cosimplicial $\mathcal{O}$-algebra is defined in terms of the associated Tot tower of $\mathcal{O}$-algebras, and the Bousfield-Kan identification of the resulting E^2 term requires the fundamental pullback diagrams constructing Tot^n from Tot^{n-1} for $\mathcal{O}$-algebras. In other words, this section makes precise the various definitions and constructions for cosimplicial $\mathcal{O}$-algebras that readers familiar with [19] will recognize as favorite tools from the context of cosimplicial pointed spaces.

Definition 8.1. A cosimplicial $\mathcal{O}$-algebra Z is *coaugmented* if it comes with a map

$$d^0\colon Z^{-1}{\rightarrow}Z^0 \tag{48}$$

of $\mathcal{O}$-algebras such that $d^0d^0=d^1d^0\colon Z^{-1}{\rightarrow}Z^1$; in this case, it follows easily from the cosimplicial identities [51, I.1] that (48) induces a map

$$Z^{-1}{\rightarrow}Z$$

of cosimplicial $\mathcal{O}$-algebras, where Z^{-1} denotes the constant cosimplicial $\mathcal{O}$-algebra with value Z^{-1}; i.e., via the inclusion $Z^{-1}\in\mathsf{Alg}_\mathcal{O}\subset(\mathsf{Alg}_\mathcal{O})^\Delta$ of constant diagrams.

We follow [27] in use of the terms *restricted cosimplicial objects* for Δ_{res}-shaped diagrams, and *restricted simplicial category* Δ_{res} to denote the subcategory of Δ with objects the totally ordered sets $[n]$ for $n \geq 0$ and morphisms the strictly monotone maps of sets $\xi\colon [n]{\rightarrow}[n']$; i.e., such that $k<l$ implies $\xi(k)<\xi(l)$.

Definition 8.2. Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules. The *totalization* functor Tot for cosimplicial $\mathcal{O}$-algebras and the *restricted totalization* (or fat totalization) functor $\mathrm{Tot}^{\mathsf{res}}$ for restricted cosimplicial $\mathcal{O}$-algebras are defined objectwise by the ends

$$\mathrm{Tot}\colon (\mathsf{Alg}_{\mathcal{O}})^{\Delta}{\rightarrow}\mathsf{Alg}_{\mathcal{O}}, \qquad X \mapsto \mathbf{hom}(\Delta[-],X)^{\Delta}$$
$$\mathrm{Tot}^{\mathsf{res}}\colon (\mathsf{Alg}_{\mathcal{O}})^{\Delta_{\mathsf{res}}}{\rightarrow}\mathsf{Alg}_{\mathcal{O}}, \qquad Y \mapsto \mathbf{hom}(\Delta[-],Y)^{\Delta_{\mathsf{res}}}$$

We often drop the adjective "restricted" and simply refer to both functors as *totalization* functors. It follows from the universal property of ends that $\mathrm{Tot}(X)$ is naturally isomorphic to an equalizer diagram of the form

$$\mathrm{Tot}(X) \cong \lim\Big(\prod_{[n]\in\Delta} \mathbf{hom}(\Delta[n],X^n) \rightrightarrows \prod_{\substack{[n]\rightarrow[n'] \\ \text{in }\Delta}} \mathbf{hom}(\Delta[n],X^{n'}) \Big)$$

in $\mathsf{Alg}_{\mathcal{O}}$, and similarly for $\mathrm{Tot}^{\mathsf{res}}(Y)$ by replacing Δ with Δ_{res}. We sometimes refer to the natural maps $\mathrm{Tot}(X){\rightarrow}\,\mathbf{hom}(\Delta[n],X^n)$ and $\mathrm{Tot}^{\mathsf{res}}(Y){\rightarrow}\,\mathbf{hom}(\Delta[n],Y^n)$ as the *projection* maps.

Proposition 8.3. *Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules. The totalization functors* Tot *and* $\mathrm{Tot}^{\mathsf{res}}$ *fit into adjunctions*

$$\mathsf{Alg}_{\mathcal{O}} \underset{\mathrm{Tot}}{\overset{-\dot\otimes\Delta[-]}{\rightleftarrows}} (\mathsf{Alg}_{\mathcal{O}})^{\Delta}\ , \qquad \mathsf{Alg}_{\mathcal{O}} \underset{\mathrm{Tot}^{\mathsf{res}}}{\overset{-\dot\otimes\Delta[-]}{\rightleftarrows}} (\mathsf{Alg}_{\mathcal{O}})^{\Delta_{\mathsf{res}}} \tag{49}$$

with left adjoints on top.

Proof. Consider the case of Tot (resp. $\mathrm{Tot}^{\mathsf{res}}$). Using the universal property of ends, it is easy to verify that the functor given objectwise by $A\dot\otimes\Delta[-]$ (resp. $A\dot\otimes\Delta[-]$) is a left adjoint of Tot (resp. $\mathrm{Tot}^{\mathsf{res}}$). ☐

Definition 8.4. Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules and D a small category. The *Bousfield-Kan homotopy limit* functor $\mathrm{holim}^{\mathsf{BK}}_{\mathsf{D}}$ for D-shaped diagrams in $\mathsf{Alg}_{\mathcal{O}}$ is defined objectwise by

$$\mathrm{holim}^{\mathsf{BK}}_{\mathsf{D}}\colon (\mathsf{Alg}_{\mathcal{O}})^{\mathsf{D}}{\rightarrow}\mathsf{Alg}_{\mathcal{O}}, \qquad X \mapsto \mathrm{Tot}\ {\textstyle\prod^*_{\mathsf{D}}}\ X$$

We will sometimes suppress D from the notation and simply write $\mathrm{holim}^{\mathsf{BK}}$ and $\prod^*$. Here, the cosimplicial replacement functor $\prod^*\colon (\mathsf{Alg}_{\mathcal{O}})^{\mathsf{D}}{\rightarrow}(\mathsf{Alg}_{\mathcal{O}})^{\Delta}$ is defined objectwise by

$${\textstyle\prod^n} X := \prod_{\substack{a_0\rightarrow\cdots\rightarrow a_n \\ \text{in }\mathsf{D}}} X(a_n)$$

with the obvious coface d^i and codegeneracy maps s^j; in other words, such that d^i "misses i" and s^j "doubles j" on the projection maps inducing these maps; compare, [19, XI.5]. For a useful introduction in the dual setting of homotopy colimits, realization, and the simplicial replacement functor, in the context of spaces, see [35].

Remark 8.5. The basic idea behind the cosimplicial replacement $\prod^*_{\mathsf{D}} X \in (\mathsf{Alg}_{\mathcal{O}})^{\Delta}$ of a D-shaped diagram X is that it arises as a natural cosimplicial resolution of $\lim_{\mathsf{D}} X$; in other words, $\lim_{\mathsf{D}} X$ is naturally isomorphic to an equalizer of the form

$$\lim_{\mathsf{D}} X \cong \lim\Big(\prod_{a_0 \in \mathsf{D}} X(a_0) \overset{d^0}{\underset{d^1}{\rightrightarrows}} \prod_{\substack{a_0 \to a_1 \\ \text{in } \mathsf{D}}} X(a_1) \Big) \cong \lim_{\Delta} \textstyle\prod^*_{\mathsf{D}} X$$

This description of $\lim_{\mathsf{D}} X$ naturally arises when verifying existence of $\lim_{\mathsf{D}} X$ in terms of existence of small products and equalizers; i.e., the description that arises by writing down the desired universal property of the limiting cone of $\lim_{\mathsf{D}} X$.

Proposition 8.6. *Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules and D a small category. Let $X \in (\mathsf{Alg}_{\mathcal{O}})^{\mathsf{D}}$. Then $\operatorname{holim}^{\mathsf{BK}}_{\mathsf{D}} X$ is naturally isomorphic to the end construction*

$$\begin{aligned} \operatorname{holim}^{\mathsf{BK}}_{\mathsf{D}} X &\cong \mathbf{hom}_{\mathsf{D}}(B(\mathsf{D}/-), X) \\ &= \mathbf{hom}(B(\mathsf{D}/-), X)^{\mathsf{D}} \end{aligned}$$

in $\mathsf{Alg}_{\mathcal{O}}$; the end construction $\mathbf{hom}_{\mathsf{D}}(B(\mathsf{D}/-), X)$ can be thought of as the mapping object of D-shaped diagrams.

Remark 8.7. Here, $\mathsf{D}/-$ denotes the over category $\mathsf{D}\downarrow -$ (or comma category) functor, $B\colon \mathsf{Cat}{\rightarrow}\mathsf{sSet}$ the nerve functor, and Cat the category of small categories (see [77, II.6] and [51, I.1.4]).

Proof of Proposition 8.6. This is proved in [19, XI.5, XI.3] in the context of spaces, and the same argument verifies it in our context. □

Proposition 8.8. *Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules and D a small category. The Bousfield-Kan homotopy limit functor $\operatorname{holim}^{\mathsf{BK}}_{\mathsf{D}}$ fits into an adjunction*

$$\mathsf{Alg}_{\mathcal{O}} \underset{\operatorname{holim}^{\mathsf{BK}}_{\mathsf{D}}}{\overset{-\dot{\otimes} B(\mathsf{D}/-)}{\rightleftarrows}} (\mathsf{Alg}_{\mathcal{O}})^{\mathsf{D}} \tag{50}$$

with left adjoint on top. Furthermore, this adjunction is a Quillen adjunction with respect to the projective model structure on D-shaped diagrams induced from $\mathsf{Alg}_{\mathcal{O}}$.

Proof. This is proved in [19, XI.3, XI.8] in the context of spaces, and the same argument verifies it in our context. For instance, to verify it is a Quillen adjunction, it suffices to verify that cosimplicial replacement $\prod^*$ sends objectwise (acyclic) fibrations in $(\mathsf{Alg}_{\mathcal{O}})^{\mathsf{D}}$ to (acyclic) Reedy fibrations in $(\mathsf{Alg}_{\mathcal{O}})^{\Delta}$; this is proved in [19, XI.5.3] (see also [65] for a useful development) and exactly the same argument verifies it in our context. □

Definition 8.9. Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules and D a small category. The $\operatorname{holim}_{\mathsf{D}}$ functor for D-shaped diagrams in $\mathsf{Alg}_{\mathcal{O}}$ is defined objectwise by

$$\operatorname{holim}_{\mathsf{D}}\colon (\mathsf{Alg}_{\mathcal{O}})^{\mathsf{D}} {\rightarrow} \mathsf{Alg}_{\mathcal{O}}, \qquad X \mapsto \operatorname{holim}^{\mathsf{BK}}_{\mathsf{D}} X^f$$

where X^f denotes a functorial fibrant replacement of X in $(\mathsf{Alg}_{\mathcal{O}})^{\mathsf{D}}$ with respect to the projective model structure on D-shaped diagrams induced from $\mathsf{Alg}_{\mathcal{O}}$.

Remark 8.10. It follows that there is a natural weak equivalence

$$\operatorname{holim}_{\mathsf{D}} X \simeq \mathsf{R} \operatorname{holim}^{\mathsf{BK}}_{\mathsf{D}} X$$

and if furthermore, X is objectwise fibrant, then the natural map

$$\mathrm{holim}^{\mathsf{BK}}_{\mathsf{D}} X \xrightarrow{\simeq} \mathrm{holim}_{\mathsf{D}} X$$

is a weak equivalence. Here, $\mathsf{R}\,\mathrm{holim}^{\mathsf{BK}}_{\mathsf{D}}$ denotes the total right derived functor of $\mathrm{holim}^{\mathsf{BK}}_{\mathsf{D}}$.

8.11. **Truncation filtration and the associated** $\mathrm{holim}^{\mathsf{BK}}$ **tower in** $\mathsf{Alg}_{\mathcal{O}}$**.** The simplicial category Δ has a natural filtration by its truncated subcategories $\Delta^{\leq n}$ of the form

$$\emptyset \subset \Delta^{\leq 0} \subset \Delta^{\leq 1} \subset \cdots \subset \Delta^{\leq n} \subset \Delta^{\leq n+1} \subset \cdots \subset \Delta$$

where $\Delta^{\leq n} \subset \Delta$ denotes the full subcategory of objects $[m]$ such that $m \leq n$; we use the convention that $\Delta^{\leq -1} = \emptyset$ is the empty category. This leads to the following $\mathrm{holim}^{\mathsf{BK}}$ tower of a cosimplicial $\mathcal{O}$-algebra.

Proposition 8.12. *Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules. If X is a cosimplicial $\mathcal{O}$-algebra, then $\mathrm{holim}^{\mathsf{BK}}_{\Delta} X$ is naturally isomorphic to a limit of the form*

$$\mathrm{holim}^{\mathsf{BK}}_{\Delta} X \cong \lim\big(*\leftarrow \mathrm{holim}^{\mathsf{BK}}_{\Delta^{\leq 0}} X \leftarrow \mathrm{holim}^{\mathsf{BK}}_{\Delta^{\leq 1}} X \leftarrow \mathrm{holim}^{\mathsf{BK}}_{\Delta^{\leq 2}} X \leftarrow \cdots\big)$$

in $\mathsf{Alg}_{\mathcal{O}}$; in particular, $\mathrm{holim}^{\mathsf{BK}}_{\Delta^{\leq -1}} X = *$ and $\mathrm{holim}^{\mathsf{BK}}_{\Delta^{\leq 0}} X \cong X^0$. We usually refer to the tower $\{\mathrm{holim}^{\mathsf{BK}}_{\Delta^{\leq s}}\}_{s\geq -1}$ as the $\mathrm{holim}^{\mathsf{BK}}$ tower of X.

Proof. We know that $\Delta \cong \mathrm{colim}_s \Delta^{\leq s}$, hence it follows that

$$\mathbf{hom}_{\Delta}(\Delta[-], \textstyle\prod^*_{\Delta} X) \cong \lim_s \mathbf{hom}_{\Delta}(\Delta[-], \textstyle\prod^*_{\Delta^{\leq s}} X) = \lim\limits_s \mathrm{holim}^{\mathsf{BK}}_{\Delta^{\leq s}} X$$

Here, we have used the natural isomorphism $\prod^*_{\Delta} X \cong \lim_s \prod^*_{\Delta^{\leq s}} X$ of cosimplicial $\mathcal{O}$-algebras. ☐

Proposition 8.13. *Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules and D a small category. If $\mathsf{D}' \subset \mathsf{D}$ is a subcategory and $X \in (\mathsf{Alg}_{\mathcal{O}})^{\mathsf{D}}$ is objectwise fibrant, then the natural map*

$$\textstyle\prod^*_{\mathsf{D}'} X \leftarrow \prod^*_{\mathsf{D}} X$$

in $(\mathsf{Alg}_{\mathcal{O}})^{\Delta}$ is a Reedy fibration.

Proof. This is closely related to [19, XI.5.2] and the same argument works in this context; compare [30, A.7.2.4]. ☐

Proposition 8.14. *Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules and D a small category. If $\mathsf{D}' \subset \mathsf{D}$ is a subcategory and $X \in (\mathsf{Alg}_{\mathcal{O}})^{\mathsf{D}}$ is objectwise fibrant, then the natural map*

$$\mathrm{holim}^{\mathsf{BK}}_{\mathsf{D}'} X \leftarrow \mathrm{holim}^{\mathsf{BK}}_{\mathsf{D}} X$$

in $\mathsf{Alg}_{\mathcal{O}}$ is a fibration.

Proof. This follows from Proposition 8.13; it also appears in [53] in the context of spaces. ☐

Proposition 8.15. *Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules.*

(a) *If $X \in (\mathsf{Alg}_{\mathcal{O}})^{\Delta}$ is objectwise fibrant, then the tower $\{\mathrm{holim}^{\mathsf{BK}}_{\Delta^{\leq s}} X\}_{s\geq -1}$ is a tower of fibrations, and the natural map $\mathrm{holim}^{\mathsf{BK}}_{\Delta} X \rightarrow \mathrm{holim}^{\mathsf{BK}}_{\Delta^{\leq s}} X$ is a fibration for each $s \geq -1$; in particular, $\mathrm{holim}^{\mathsf{BK}}_{\Delta} X$ is fibrant.*

(b) *If $X \rightarrow X'$ in $(\mathsf{Alg}_{\mathcal{O}})^{\Delta}$ is a weak equivalence between objectwise fibrant objects, then $\operatorname{holim}^{\mathsf{BK}}_{\Delta} X \rightarrow \operatorname{holim}^{\mathsf{BK}}_{\Delta} X'$ is a weak equivalence.*

Proof. Part (a) follows from Proposition 8.14 and part (b) follows from the Quillen adjunction (50); see, for instance, [36, 9.8, 9.9]; it can also be argued exactly as in [19, XI.5.4]. □

The following proposition is proved in [27, 3.17].

Proposition 8.16. *The inclusion of categories $\Delta_{\mathsf{res}} \subset \Delta$ is left cofinal (i.e., homotopy initial).*

Proposition 8.17. *If $X \in (\mathsf{Alg}_{\mathcal{O}})^{\Delta}$ is objectwise fibrant and $Y \in (\mathsf{Alg}_{\mathcal{O}})^{\Delta}$ is Reedy fibrant, then the natural maps*

$$\operatorname{holim}^{\mathsf{BK}}_{\Delta} X \xrightarrow{\simeq} \operatorname{holim}^{\mathsf{BK}}_{\Delta_{\mathsf{res}}} X \xleftarrow{\simeq} \operatorname{Tot}^{\mathsf{res}} X, \qquad \operatorname{holim}^{\mathsf{BK}}_{\Delta} Y \xleftarrow{\simeq} \operatorname{Tot} Y,$$

in $\mathsf{Alg}_{\mathcal{O}}$ are weak equivalences.

Proof. This follows from left cofinality (Proposition 8.16) and [19] in the context of spaces, and exactly the same argument verifies it in this context. □

8.18. **Skeletal filtration and the associated** Tot **towers in $\mathsf{Alg}_{\mathcal{O}}$.**

Definition 8.19. Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules and $s \geq -1$. The functors Tot_s and $\operatorname{Tot}^{\mathsf{res}}_s$ are defined objectwise by the ends

$$\operatorname{Tot}_s\colon (\mathsf{Alg}_{\mathcal{O}})^{\Delta} \rightarrow \mathsf{Alg}_{\mathcal{O}}, \qquad X \mapsto \mathbf{hom}(\operatorname{sk}_s\Delta[-], X)^{\Delta}$$
$$\operatorname{Tot}^{\mathsf{res}}_s\colon (\mathsf{Alg}_{\mathcal{O}})^{\Delta_{\mathsf{res}}} \rightarrow \mathsf{Alg}_{\mathcal{O}}, \qquad Y \mapsto \mathbf{hom}(\operatorname{sk}_s\Delta[-], Y)^{\Delta_{\mathsf{res}}}$$

Here we use the convention that the (-1)-skeleton of a simplicial set is the empty simplicial set. In particular, $\operatorname{sk}_{-1}\Delta[n] = \emptyset$ for each $n \geq 0$; it follows immediately that $\operatorname{Tot}_{-1}(X) = *$ and $\operatorname{Tot}^{\mathsf{res}}_{-1}(Y) = *$.

Proposition 8.20. *Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules and $s \geq 0$. The functors Tot_s and $\operatorname{Tot}^{\mathsf{res}}_s$ fit into adjunctions*

$$\mathsf{Alg}_{\mathcal{O}} \underset{\operatorname{Tot}_s}{\overset{-\dot{\otimes}\operatorname{sk}_s\Delta[-]}{\rightleftarrows}} (\mathsf{Alg}_{\mathcal{O}})^{\Delta}, \qquad \mathsf{Alg}_{\mathcal{O}} \underset{\operatorname{Tot}^{\mathsf{res}}_s}{\overset{-\dot{\otimes}\operatorname{sk}_s\Delta[-]}{\rightleftarrows}} (\mathsf{Alg}_{\mathcal{O}})^{\Delta_{\mathsf{res}}}$$

with left adjoints on top. It follows that there are natural isomorphisms $\operatorname{Tot}_0(X) \cong X^0$ and $\operatorname{Tot}^{\mathsf{res}}_0(Y) \cong Y^0$.

Proof. The first part follows exactly as in the proof of Proposition 8.3. The second part follows from uniqueness of right adjoints, up to isomorphism. For instance, to verify the natural isomorphism $\operatorname{Tot}_0(X) \cong X^0$, it suffices to verify that giving a map $A \rightarrow X^0$ in $\mathsf{Alg}_{\mathcal{O}}$ is the same as giving a map $A \dot{\otimes} \operatorname{sk}_0\Delta[-] \rightarrow X$ in $(\mathsf{Alg}_{\mathcal{O}})^{\Delta}$; this follows from the universal property of ends together with the natural isomorphisms

$$A \dot{\otimes} \operatorname{sk}_0\Delta[n] \cong \coprod_{\substack{[0]\rightarrow[n] \\ \text{in } \Delta}} A$$

in $\mathsf{Alg}_{\mathcal{O}}$. The case of $\operatorname{Tot}^{\mathsf{res}}_0(Y) \cong Y$ is similar by replacing Δ with Δ_{res}. □

The following can be thought of as the analog of the $\operatorname{holim}^{\mathsf{BK}}$ tower of a cosimplicial $\mathcal{O}$-algebra; under appropriate fibrancy conditions these towers are naturally weakly equivalent.

Proposition 8.21. *Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules. Let X be a cosimplicial (resp. restricted cosimplicial) $\mathcal{O}$-algebra. The totalization* $\mathrm{Tot}(X)$ *(resp.* $\mathrm{Tot}^{\mathsf{res}}(X)$*) is naturally isomorphic to a limit of the form*

$$\mathrm{Tot}(X) \cong \lim\big(*\leftarrow \mathrm{Tot}_0(X)\leftarrow \mathrm{Tot}_1(X)\leftarrow \mathrm{Tot}_2(X)\leftarrow\cdots\big)$$

$$\textit{resp.}\quad \mathrm{Tot}^{\mathsf{res}}(X) \cong \lim\Big(*\leftarrow \mathrm{Tot}^{\mathsf{res}}_0(X)\leftarrow \mathrm{Tot}^{\mathsf{res}}_1(X)\leftarrow \mathrm{Tot}^{\mathsf{res}}_2(X)\leftarrow\cdots\Big)$$

in $\mathsf{Alg}_{\mathcal{O}}$*. We often refer to the tower* $\{\mathrm{Tot}_s(X)\}_{s\geq -1}$ *(resp.* $\{\mathrm{Tot}^{\mathsf{res}}_s(X)\}_{s\geq -1}$*) as the* Tot *tower (resp.* $\mathrm{Tot}^{\mathsf{res}}$ *tower) of* X *.*

Proof. We know that $\Delta[-] \cong \mathrm{colim}_s\, \mathrm{sk}_s\Delta[-]$ in sSet^{Δ}. Since the contravariant functor $\mathbf{hom}(-,X)^{\Delta}\colon \mathsf{sSet}^{\Delta}\rightarrow\mathsf{Alg}_{\mathcal{O}}$ sends colimiting cones to limiting cones (see Remark 7.5), it follows that there are natural isomorphisms

$$\mathbf{hom}(\Delta[-],X)^{\Delta} \cong \mathbf{hom}(\mathrm{colim}_s\, \mathrm{sk}_s\Delta[-],X)^{\Delta} \cong \lim_s \mathbf{hom}(\mathrm{sk}_s\Delta[-],X)^{\Delta}$$

which finishes the proof for the case of $\mathrm{Tot}(X)$. The case of $\mathrm{Tot}^{\mathsf{res}}(X)$ is similar by replacing Δ with Δ_{res}. □

The following two propositions, which construct Tot_s from Tot_{s-1} (resp. $\mathrm{Tot}^{\mathsf{res}}_s$ from $\mathrm{Tot}^{\mathsf{res}}_{s-1}$), play a key role in the homotopical analysis of the totalization functors; it is useful to contrast the two pullback constructions as a helpful way to understand the difference between Tot and $\mathrm{Tot}^{\mathsf{res}}$.

Proposition 8.22. *Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules. Let X be a cosimplicial $\mathcal{O}$-algebra and $s\geq 0$. There is a pullback diagram of the form*

$$\begin{array}{ccc}
\mathrm{Tot}_s(X) & \longrightarrow & \mathbf{hom}(\Delta[s],X^s) \\
\downarrow & & \downarrow \\
\mathrm{Tot}_{s-1}(X) & \longrightarrow & \mathbf{hom}(\partial\Delta[s],X^s)\times_{\mathbf{hom}(\partial\Delta[s],M^{s-1}X)}\mathbf{hom}(\Delta[s],M^{s-1}X)
\end{array}$$

in $\mathsf{Alg}_{\mathcal{O}}$*. Here, we are using the matching object notation $M^{s-1}X$ for a cosimplicial object (see* [51, VIII.1]*); in particular, $M^{-1}X = *$ and $M^0X\cong X^0$.*

Proof. This follows from [51, VIII.1]. □

Proposition 8.23. *Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules. Let Y be a restricted cosimplicial $\mathcal{O}$-algebra and $s\geq 0$. There is a pullback diagram of the form*

$$\begin{array}{ccc}
\mathrm{Tot}^{\mathsf{res}}_s(Y) & \longrightarrow & \mathbf{hom}(\Delta[s],Y^s) \\
\downarrow & & \downarrow \\
\mathrm{Tot}^{\mathsf{res}}_{s-1}(Y) & \longrightarrow & \mathbf{hom}(\partial\Delta[s],Y^s)
\end{array}$$

in $\mathsf{Alg}_{\mathcal{O}}$*.*

Proof. This follows from [51, VIII.1] because of the following: by uniqueness of right adjoints, up to isomorphism, $\mathrm{Tot}^{\mathsf{res}}(Y)$ is naturally isomorphic to Tot composed with the right Kan extension of Y along the inclusion $\Delta_{\mathsf{res}}\subset\Delta$. □

Remark 8.24. In the special case of $s=0$, note that Propositions 8.22 and 8.23 are simply the assertions that the natural maps

$$\mathrm{Tot}_0(X)\rightarrow \mathbf{hom}(\Delta[0],X^0)\cong X^0,\qquad \mathrm{Tot}^{\mathsf{res}}_0(Y)\rightarrow \mathbf{hom}(\Delta[0],Y^0)\cong Y^0,$$

are isomorphisms (see Proposition 8.20); this is because it follows from our conventions (see Definition 8.19) that $\mathrm{Tot}_{-1}(X) = *$ and $\mathrm{Tot}^{\mathsf{res}}_{-1}(Y) = *$.

The following proposition serves to contrast the properties of Tot and $\mathrm{Tot}^{\mathsf{res}}$ in the context of $\mathcal{O}$-algebras.

Proposition 8.25. *Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules.*

(a) *If $X \in (\mathsf{Alg}_{\mathcal{O}})^{\Delta}$ is Reedy fibrant, then the* Tot *tower $\{\mathrm{Tot}_s(X)\}$ is a tower of fibrations, and the natural map $\mathrm{Tot}(X) \rightarrow \mathrm{Tot}_s(X)$ is a fibration for each $s \geq -1$; in particular, $\mathrm{Tot}(X)$ is fibrant.*
(b) *If $Y \in (\mathsf{Alg}_{\mathcal{O}})^{\Delta_{\mathsf{res}}}$ is objectwise fibrant, then the* $\mathrm{Tot}^{\mathsf{res}}$ *tower $\{\mathrm{Tot}^{\mathsf{res}}_s(Y)\}$ is a tower of fibrations, and the natural map $\mathrm{Tot}^{\mathsf{res}}(Y) \rightarrow \mathrm{Tot}^{\mathsf{res}}_s(Y)$ is a fibration for each $s \geq -1$; in particular, $\mathrm{Tot}^{\mathsf{res}}(Y)$ is fibrant.*
(c) *If $X \rightarrow X'$ in $(\mathsf{Alg}_{\mathcal{O}})^{\Delta}$ is a weak equivalence between Reedy fibrant objects, then $\mathrm{Tot}(X) \rightarrow \mathrm{Tot}(X')$ is a weak equivalence.*
(d) *If $Y \rightarrow Y'$ in $(\mathsf{Alg}_{\mathcal{O}})^{\Delta_{\mathsf{res}}}$ is a weak equivalence between objectwise fibrant diagrams, then $\mathrm{Tot}^{\mathsf{res}}(Y) \rightarrow \mathrm{Tot}^{\mathsf{res}}(Y')$ is a weak equivalence.*

Proof. Consider part (a). By definition, X is Reedy fibrant if the natural map $X^s \rightarrow M^{s-1}X$ is a fibration for each $s \geq 0$. It follows from the pullback diagrams in Proposition 8.22 that each map $\mathrm{Tot}_s(X) \rightarrow \mathrm{Tot}_{s-1}(X)$ is a fibration, and hence each natural map $\mathrm{Tot}(X) \rightarrow \mathrm{Tot}_s(X)$ is a fibration. Part (b) follows similarly from the pullback diagrams in Proposition 8.23. Consider part (c). Since $X \rightarrow X'$ is a weak equivalence between Reedy fibrant objects, it follows from [51, VIII.1] that $M^{s-1}X \rightarrow M^{s-1}X'$ is a weak equivalence between fibrant objects. Hence by Proposition 8.22 the induced map $\{\mathrm{Tot}_s X\} \rightarrow \{\mathrm{Tot}_s X'\}$ is an objectwise weak equivalence between towers of fibrations, and applying the limit functor $\lim_s$ finishes the proof that $\mathrm{Tot}(X) \rightarrow \mathrm{Tot}(X')$ is a weak equivalence. Part (d) follows similarly by using Proposition 8.23 instead of Proposition 8.22. □

8.26. **Comparing the** $\mathrm{holim}^{\mathsf{BK}}$ **and** Tot **towers in** $\mathsf{Alg}_{\mathcal{O}}$**.** The following is proved in Carlsson [22, Section 6], Dugger [28], and Sinha [106, 6.7]; see also Dundas-Goodwillie-McCarthy [30] and Munson-Volic [89]. This cofinality result is implicit in the early work of Hopkins [62].

Proposition 8.27. *Let $n \geq 0$. The composite*

$$\mathcal{P}_0([n]) \cong P\Delta[n] \longrightarrow \Delta^{\leq n}_{\mathsf{res}} \subset \Delta^{\leq n}$$

is left cofinal (i.e., homotopy initial). Here, $\mathcal{P}_0([n])$ denotes the poset of all nonempty subsets of $[n]$; this notation agrees with [53] *and* [25, 3.1].

Remark 8.28. Note that $\mathcal{P}_0([n]) \cong P\Delta[n]$ naturally arises as the category whose nerve is the subdivision $\mathrm{sd}\Delta[n]$ of $\Delta[n]$, and that applying realization recovers the usual barycentric subdivision of Δ^n (see [51, III.4]).

Remark 8.29. One can ask if left cofinality of the inclusion $\Delta_{\mathsf{res}} \subset \Delta$ (Proposition 8.16) remains true when Δ is replaced by its n-truncation $\Delta^{\leq n}$. It turns out that the inclusion $\Delta^{\leq n}_{\mathsf{res}} \subset \Delta^{\leq n}$ is not left cofinal for $n \geq 1$; Proposition 8.27 can be thought of as the n-truncated analog of Proposition 8.16. The upshot is that if one further precomposes to the poset $P\Delta[n]$ of non-degenerate simplices of $\Delta[n]$, ordered by the face relation (see [51, III.4]), then the resulting composite $P\Delta[n] \rightarrow \Delta^{\leq n}$ is left cofinal.

Proposition 8.30. *Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules. If $X \in (\mathsf{Alg}_{\mathcal{O}})^{\Delta}$ is objectwise fibrant and $Y \in (\mathsf{Alg}_{\mathcal{O}})^{\Delta}$ is Reedy fibrant, then the natural maps*

$$\operatorname{holim}^{\mathsf{BK}}_{\Delta^{\leq n}} X \xrightarrow{\simeq} \operatorname{holim}^{\mathsf{BK}}_{P\Delta[n]} X \cong \operatorname{holim}^{\mathsf{BK}}_{\mathcal{P}_0([n])} X$$
$$\operatorname{Tot}_n Y \xrightarrow{\simeq} \operatorname{holim}^{\mathsf{BK}}_{\Delta^{\leq n}} Y$$

in $\mathsf{Alg}_{\mathcal{O}}$ are weak equivalences.

Proof. This appears in [22] and [106] in the contexts of spectra and spaces, respectively, and remains true in this context. The first case follows from Proposition 8.27 and [19], and the second case follows from [19], where the indicated natural weak equivalence is the composite

$$\begin{aligned}\mathbf{hom}_{\Delta}(\operatorname{sk}_n \Delta[-], Y) &\cong \mathbf{hom}_{\Delta^{\leq n}}(\Delta^{\leq n}[-], Y)\\ &\xrightarrow{\simeq} \mathbf{hom}_{\Delta^{\leq n}}(B(\Delta^{\leq n}/-), Y) \cong \operatorname{holim}^{\mathsf{BK}}_{\Delta^{\leq n}} Y\end{aligned}$$

in $\mathsf{Alg}_{\mathcal{O}}$; note that the natural map $B(\Delta^{\leq n}/-) \xrightarrow{\simeq} \Delta^{\leq n}[-]$ in $(\mathsf{sSet})^{\Delta^{\leq n}}$ is a weak equivalence between Reedy cofibrant objects. ☐

8.31. **Homotopy spectral sequence of a tower of $\mathcal{O}$-algebras.** Consider any tower $\{X_s\}$ of fibrations of $\mathcal{O}$-algebras such that $X_{-1} = *$ and denote by $F_s \subset X_s$ the fiber of $X_s \rightarrow X_{s-1}$, where $F_0 = X_0$.

Remark 8.32. For ease of notational purposes, we will regard such towers as indexed by the integers such that $X_s = *$ (and hence $F_s = *$) for every $s < 0$.

Following the notation in [19, IX.4] as closely as possible, recall that the collection of *r-th derived long exact sequences*, $r \geq 0$, associated to the collection of homotopy fiber sequences $F_s \to X_s \to X_{s-1}$, $s \in \mathbb{Z}$, has the form

$$\begin{gathered}\cdots \rightarrow \pi_{i+1} F^{(r)}_{s-r} \rightarrow \pi_{i+1} X^{(r)}_{s-r} \rightarrow \pi_{i+1} X^{(r)}_{s-r-1} \rightarrow\\ \pi_i F^{(r)}_{s} \rightarrow \pi_i X^{(r)}_{s} \rightarrow \pi_i X^{(r)}_{s-1} \rightarrow\\ \pi_{i-1} F^{(r)}_{s+r} \rightarrow \pi_{i-1} X^{(r)}_{s+r} \rightarrow \pi_{i-1} X^{(r)}_{s+r-1} \rightarrow \cdots\end{gathered} \tag{51}$$

where we define

$$\pi_i X^{(r)}_s := \operatorname{im}(\pi_i X_s \leftarrow \pi_i X_{s+r}), \qquad r \geq 0, \tag{52}$$

$$\pi_i F^{(r)}_s := \frac{\ker(\pi_i F_s \rightarrow \pi_i X_s / \pi_i X^{(r)}_s)}{\partial_* \ker(\pi_{i+1} X_{s-1} \rightarrow \pi_{i+1} X_{s-(r+1)})}, \qquad r \geq 0, \tag{53}$$

and $\pi_i X^{(0)}_s = \pi_i X_s$ and $\pi_i F^{(0)}_s = \pi_i F_s$, for each $i \in \mathbb{Z}$. Here, we denote by ∂ the natural boundary maps appearing in the 0-th derived long exact sequences (i.e., the long exact sequences in π_*) associated to each $F_s \rightarrow X_s \rightarrow X_{s-1}$.

In other words, the collection of long exact sequences in π_* associated to the collection of homotopy fiber sequences $F_s \rightarrow X_s \rightarrow X_{s-1}$, $s \in \mathbb{Z}$, gives an exact couple $(\pi_* X_*, \pi_* F_*)$ of $\mathbb{Z}$-bigraded abelian groups of the form

$$\begin{array}{ccc} \pi_* X_* & \longrightarrow & \pi_* X_* \\ & \nwarrow & \downarrow \\ & & \pi_* F_* \end{array} \qquad \begin{array}{ccc} D & \xrightarrow{(1,-1)} & D \\ & \nwarrow^{(-1,0)} & \downarrow{\scriptstyle (0,0)} \\ & & E \end{array} \tag{54}$$

If we denote this exact couple by (D,E), with bigradings defined by $E_{-s,t} := \pi_{t-s}F_s$ and $D_{-s,t} := \pi_{t-(s-1)}X_{s-1}$, then it is easy to verify that the three maps depicted on the right-hand side of (54) have the indicated bidegrees. It follows that the associated collection of r-th derived exact couples, $r \geq 0$, determines a left-half plane homologically graded spectral sequence (E^r, d^r), $r \geq 1$ (see [76, XI.5] for a useful development); in particular, d^r has bidegree $(-r, r-1)$. We will sometimes denote E^r by $E^r\{X_s\}$ to emphasize the tower $\{X_s\}$ in the notation.

This is the *homotopy spectral sequence of the tower of fibrations* $\{X_s\}$ of $\mathcal{O}$-algebras and satisfies

$$E^r_{-s,s+i} := \pi_i F_s^{(r-1)}, \qquad \big(\text{equiv.} \qquad E^r_{-s,t} := \pi_{t-s} F_s^{(r-1)}\big), \qquad r \geq 1, \tag{55}$$

with differentials $d^r\colon E^r_{-s,s+i} \to E^r_{-(s+r),s+r+i-1}$ given by the composite maps

$$\pi_i F_s^{(r-1)} \to \pi_i X_s^{(r-1)} \to \pi_{i-1} F_{s+r}^{(r-1)}. \tag{56}$$

It is essentially identical to the homotopy spectral sequence of a tower of fibrations in pointed spaces described in [19], except it is better behaved in the sense that it has no "fringing" (e.g., [18], [51, VI.2]). The fringing issue arises in spaces because the long exact sequence of homotopy groups (associated to a fibration sequence) is not an exact sequence of abelian groups in low dimensions; this difficulty is not present for fibration sequences of spectra or $\mathcal{O}$-algebras, and hence no fringing issues arise here.

Definition 8.33. Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules. Suppose $\{X_s\}$ is a tower in $\mathsf{Alg}_{\mathcal{O}}$. The *homotopy spectral sequence* (E^r, d^r)*,* $r \geq 1$*, associated to the tower* $\{X_s\}$ is the homotopy spectral sequence of the functorial fibrant replacement $\{X_s\}^f$ of $\{X_s\}$ in the category of towers in $\mathsf{Alg}_{\mathcal{O}}$. We sometimes denote E^r by $E^r\{X_s\}$ to emphasize in the notation the tower $\{X_s\}$.

The following Milnor-type short exact sequences are well known in stable homotopy theory (see, for instance, [34]); they can be established as a consequence of [19, IX].

Proposition 8.34. *Consider any tower* $B_0 \leftarrow B_1 \leftarrow B_2 \leftarrow \cdots$ *of* $\mathcal{O}$*-algebras. There are natural short exact sequences*

$$0 \to \lim^1_k \pi_{i+1} B_k \to \pi_i \operatorname*{holim}_k B_k \to \lim_k \pi_i B_k \to 0.$$

The following is a spectral algebra analog of the Bousfield-Kan connectivity lemma [19, IX.5.1] for spaces.

Proposition 8.35. *Let* $\mathcal{O}$ *be an operad in* $\mathcal{R}$*-modules. Suppose* $\{X_s\}$ *is a tower in* $\mathsf{Alg}_{\mathcal{O}}$*. Let* $n \in \mathbb{Z}$ *and* $r \geq 1$*. If* $E^r_{-s,s+i} = 0$ *for each* $i \leq n$ *and* s*, then*

(a) $\operatorname{holim}_s X_s$ *is* n*-connected,*
(b) $\lim_s \pi_i X_s = 0 = \lim^1_s \pi_{i+1} X_s$ *for each* $i \leq n$*.*

Proof. This is proved in [19, IX.5.1] in the context of pointed spaces, and exactly the same argument verifies it in our context. □

The following is a spectral algebra analog of the Bousfield-Kan mapping lemma [19, IX.5.2] for spaces.

Proposition 8.36. *Let* $\mathcal{O}$ *be an operad in* $\mathcal{R}$*-modules. Let* $f\colon \{X_s\} \to \{X'_s\}$ *be a map of towers in* $\mathsf{Alg}_{\mathcal{O}}$*. Let* $r \geq 1$*. If* f *induces an* E^r*-isomorphism* $E^r\{X_s\} \cong E^r\{X'_s\}$ *between homotopy spectral sequences, then*

(a) *f induces a weak equivalence* $\operatorname{holim}_s X_s \simeq \operatorname{holim}_s X'_s$,
(b) *f induces isomorphisms*

$$\lim_s \pi_i X_s \cong \lim_s \pi_i X'_s, \qquad \lim^1_s \pi_i X_s \cong \lim^1_s \pi_i X'_s, \qquad i \in \mathbb{Z},$$

(c) *f induces a pro-isomorphism* $\{\pi_i X_s\} \to \{\pi_i X'_s\}$ *for each* $i \in \mathbb{Z}$.

Proof. This is proved in [19, IX.5.2] in the context of pointed spaces, and exactly the same argument verifies it in our context. □

The following can be thought of as a relative connectivity lemma for towers of $\mathcal{O}$-algebras; it is closely related to [19, I.6.2, IV.5.1] in the context of spaces.

Proposition 8.37. *Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules. Let $f\colon \{X_s\} \to \{X'_s\}$ be a map of towers in $\mathsf{Alg}_{\mathcal{O}}$. Let $n \in \mathbb{Z}$ and $r \geq 1$. Assume that f induces isomorphisms $E^r_{-s,s+i} \cong {E'}^r_{-s,s+i}$ for each $i \leq n-1$ and s, and surjections $E^r_{-s,s+i} \to {E'}^r_{-s,s+i}$ for $i = n$ and each s. Then*

(a) *f induces an $(n-1)$-connected map* $\operatorname{holim}_s X_s \to \operatorname{holim}_s X'_s$,
(b) *f induces a surjection* $\lim^1_s \pi_n X_s \to \lim^1_s \pi_n X'_s$ *and isomorphisms*

$$\lim_s \pi_i X_s \cong \lim_s \pi_i X'_s, \qquad \lim^1_s \pi_i X_s \cong \lim^1_s \pi_i X'_s, \qquad i \leq n-1.$$

If furthermore, the towers $\{\pi_n X_s\}, \{\pi_n X'_s\}, \{\pi_{n+1} X'_s\}$ are pro-constant, then

(c) *f induces an n-connected map* $\operatorname{holim}_s X_s \to \operatorname{holim}_s X'_s$,
(d) *f induces a surjection* $\lim^1_s \pi_{n+1} X_s \to \lim^1_s \pi_{n+1} X'_s$ *and isomorphisms*

$$\lim_s \pi_i X_s \cong \lim_s \pi_i X'_s, \qquad \lim^1_s \pi_i X_s \cong \lim^1_s \pi_i X'_s, \qquad i \leq n.$$

Here, ${E'}^r$ denotes $E^r\{X'_s\}$ and $\operatorname{holim}_s X_s \to \operatorname{holim}_s X'_s$ *denotes the induced zigzag in the category of $\mathcal{O}$-algebras with all backward facing maps weak equivalences.*

Proof. This is the spectral algebra analog of the argument that underlies [19, I.6.2] and [19, IV.5.1]; it follows from [19, III.2] and [19, IX]. □

9. Appendix: Totalization and $\operatorname{holim}_\Delta$ commute with realization

The purpose of this appendix is to prove Propositions 4.9 and 4.10; we describe a concise proof suggested to us by Dwyer [33]. Recall that the realization functor fits into an adjunction

$$\mathsf{sSet} \underset{\text{Sing}}{\overset{|-|}{\rightleftarrows}} \mathsf{CGHaus}, \tag{57}$$

with left adjoint on top and right adjoint the singular simplicial set functor defined objectwise by $\operatorname{Sing} Y := \hom_{\mathsf{CGHaus}}(\Delta^{(-)}, Y)$; see, for instance, [51, I.1, II.3]. In addition to the fact that (57) is a Quillen equivalence (see, for instance, [51]), the following properties of the realization functor will also be important (see [44] and [51]).

Proposition 9.1. *The realization functor* $|-|\colon \mathsf{sSet} \to \mathsf{CGHaus}$

(a) *commutes with finite limits,*
(b) *preserves fibrations,*
(c) *preserves weak equivalences.*

Proposition 9.2. *If $Y \in (\mathsf{sSet})^\Delta$ is Reedy fibrant, then $|Y| \in (\mathsf{CGHaus})^\Delta$ is Reedy fibrant.*

Proof. By assumption the canonical map $Y^{s+1} \rightarrow M^s Y$ into the indicated matching object is a fibration for each $s \geq -1$. Since realization preserves finite limits and fibrations (Proposition 9.1), it follows that $|M^s Y| \cong M^s |Y|$ and the natural map $|Y|^{s+1} \rightarrow M^s |Y|$ is a fibration for each $s \geq -1$ which finishes the proof. □

Proposition 9.3. *Let $Z \in (\mathsf{CGHaus})^{\Delta}$. There is a natural isomorphism*

$$\operatorname{Tot} \operatorname{Sing} Z \cong \operatorname{Sing} \operatorname{Tot} Z$$

Proof. Consider the following diagram of adjunctions

$$\begin{array}{ccc} (\mathsf{sSet})^{\Delta} & \underset{\text{Sing}}{\overset{|-|}{\rightleftarrows}} & (\mathsf{CGHaus})^{\Delta} \\ {\scriptstyle -\times\Delta[-]}\uparrow\downarrow{\scriptstyle \text{Tot}} & & {\scriptstyle -\times\Delta^{(-)}}\uparrow\downarrow{\scriptstyle \text{Tot}} \\ \mathsf{sSet} & \underset{\text{Sing}}{\overset{|-|}{\rightleftarrows}} & \mathsf{CGHaus} \end{array}$$

with left adjoints on top and on the left. Since $|X \times \Delta[-]| \cong |X| \times \Delta^{(-)}$, this diagram commutes up to natural isomorphism, and hence uniqueness of right adjoints (up to isomorphism) finishes the proof. □

Proposition 9.4. *If $Z \in (\mathsf{CGHaus})^{\Delta}$ is Reedy fibrant, then* $\operatorname{Sing} Z \in (\mathsf{sSet})^{\Delta}$ *is Reedy fibrant.*

Proof. Arguing exactly as in the proof of Proposition 9.2, this is because Sing preserves limits and fibrations. □

Proposition 9.5. *If $Y \in (\mathsf{sSet})^{\Delta}$ is Reedy fibrant, then* $\operatorname{Sing} |Y| \in (\mathsf{sSet})^{\Delta}$ *is Reedy fibrant.*

Proof. This follows from Propositions 9.2 and 9.4. □

Proposition 9.6. *If $Y \in (\mathsf{sSet})^{\Delta}$ is Reedy fibrant, then the natural maps*

$$\operatorname{Tot} Y \xrightarrow{\simeq} \operatorname{Tot} \operatorname{Sing} |Y| \tag{58}$$

$$|\operatorname{Tot} Y| \xrightarrow{\simeq} |\operatorname{Tot} \operatorname{Sing}(|Y|)| \tag{59}$$

$$|\operatorname{Sing} \operatorname{Tot}(|Y|)| \xrightarrow{\simeq} \operatorname{Tot} |Y| \tag{60}$$

are weak equivalences.

Proof. We know that the natural map $Y \rightarrow \operatorname{Sing} |Y|$ is a weak equivalence, since (57) is a Quillen equivalence. Since Tot sends weak equivalences between Reedy fibrant objects to weak equivalences, Proposition 9.5 verifies that (58) is a weak equivalence. It follows from (58) that the map (59) is a weak equivalence since realization preserves weak equivalences. Finally, since $|Y|$ is Reedy fibrant (Proposition 9.2), we know that $\operatorname{Tot} |Y|$ is fibrant, and hence it follows that (60) is a weak equivalence since (57) is a Quillen equivalence. □

Proof of Proposition 4.9. Consider the commutative diagram

$$\begin{array}{ccc} |\operatorname{Tot} Y| & \longrightarrow & \operatorname{Tot} |Y| \\ \downarrow \simeq & & \uparrow \simeq \\ |\operatorname{Tot} \operatorname{Sing}(|Y|)| & \xrightarrow{\cong} & |\operatorname{Sing} \operatorname{Tot}(|Y|)| \end{array}$$

By Propositions 9.6 and 9.3 the vertical maps are weak equivalences and the bottom map is an isomorphism, hence the top map is a weak equivalence. A similar argument verifies the $\mathrm{Tot}^{\mathsf{res}}$ case; by replacing Reedy fibrant with objectwise fibrant everywhere. □

Proposition 9.7. *If $Y \in (\mathsf{sSet})^{\Delta}$ is objectwise fibrant, then the cosimplicial replacements $\prod^* Y \in (\mathsf{sSet})^{\Delta}$ and $\prod^* |Y| \in (\mathsf{CGHaus})^{\Delta}$ are Reedy fibrant.*

Proof. This is straightforward. □

Proposition 9.8. *If $Y \in (\mathsf{sSet})^{\Delta}$ is objectwise fibrant, then the natural map*

$$|\prod^* Y| \xrightarrow{\simeq} \prod^* |Y|$$

is a weak equivalence.

Proof. This follows by arguing exactly as in the proof of Proposition 4.9 above. □

Proof of Proposition 4.10. The map (21) factors as

$$|\,\mathrm{Tot}\prod^* Y| \rightarrow \mathrm{Tot}\,|\prod^* Y| \rightarrow \mathrm{Tot}\prod^* |Y|$$

The left-hand map is a weak equivalence by Propositions 9.7 and 4.9, and the right-hand map is a weak equivalence by Propositions 9.7 and 9.8, since Tot sends weak equivalences between Reedy fibrant objects to weak equivalences; hence the composition is a weak equivalence. □

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