DUAL TANGENT STRUCTURES FOR ∞-TOPOSES

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Abstract. We describe dual notions of tangent bundle for an ∞-topos, each underlying a tangent ∞-category in the sense of Bauer, Burke and the author. One of those notions is Lurie's tangent bundle functor for presentable ∞-categories, and the other is its adjoint. We calculate that adjoint for injective ∞-toposes, where it is given by applying Lurie's tangent bundle on ∞-categories of points.

In [BBC21], Bauer, Burke, and the author introduce a notion of tangent ∞-category which generalizes the Rosický [Ros84] and Cockett-Cruttwell [CC14] axiomatization of the tangent bundle functor on the category of smooth manifolds and smooth maps. We also constructed there a significant example: the Goodwillie tangent structure on the ∞-category $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ of (differentiable) ∞-categories, which is built on Lurie's tangent bundle functor. That tangent structure encodes the ideas of Goodwillie's calculus of functors [Goo03] and highlights the analogy between that theory and the ordinary differential calculus of smooth manifolds.

The goal of this note is to introduce two further examples of tangent ∞-categories: one on the ∞-category of ∞-toposes and geometric morphisms, which we denote $\mathbb{T}\mathrm{opos}_\infty$, and one on the opposite ∞-category $\mathbb{T}\mathrm{opos}_\infty^{op}$ which, following Anel and Joyal [AJ19], we denote $\mathbb{L}\mathrm{ogos}_\infty$.

These two tangent structures each encode a notion of tangent bundle for ∞-toposes, but from dual perspectives. As described in [AJ19, Sec. 4], one can view ∞-toposes either from a 'geometric' or 'algebraic' point of view. In the former, an ∞-topos is thought of as a *generalized topological space*; for example, each actual topological space gives rise to an ∞-topos $\mathcal{S}h(X)$ of sheaves on X (with values in the ∞-category $\mathcal{S}$ of ∞-groupoids), and each continuous map $X \to Y$ determines a geometric morphism $\mathcal{S}h(X) \to \mathcal{S}h(Y)$. From the algebraic perspective, an ∞-topos is more like a *category of (higher) groupoids* with the ∞-topos $\mathcal{S}$ being a prime example. The 'algebraic' morphisms between two ∞-toposes are those that preserve colimits and finite limits; i.e. the left adjoints of the geometric morphisms.

Our tangent structure on the 'algebraic' ∞-category $\mathbb{L}\mathrm{ogos}_\infty$ is simply the restriction of the Goodwillie tangent structure. For an ∞-topos $\mathcal{X}$, the tangent bundle $T(\mathcal{X})$ is described by Lurie in [Lur17, 7.3.1]; the fibre of that bundle over each object $C \in \mathcal{X}$, i.e. the tangent space $T_C\mathcal{X}$, is the stabilization $\mathcal{S}p(\mathcal{X}_{/C})$ of the slice ∞-topos over C.

The tangent structure on the 'geometric' category $\mathbb{T}\mathrm{opos}_\infty$ is *dual* to that on $\mathbb{L}\mathrm{ogos}_\infty \simeq \mathbb{T}\mathrm{opos}_\infty^{op}$ in a sense described by Cockett and Cruttwell [CC14, 5.17]. Lurie's tangent bundle functor $T : \mathbb{T}\mathrm{opos}_\infty^{op} \to \mathbb{T}\mathrm{opos}_\infty^{op}$ has a left adjoint U^{op} whose opposite $U : \mathbb{T}\mathrm{opos}_\infty \to \mathbb{T}\mathrm{opos}_\infty$ is the underlying functor for a tangent structure. We call that structure the *geometric tangent structure* on $\mathbb{T}\mathrm{opos}_\infty$.

The geometric tangent structure is *representable* in the sense of [CC14, Sec. 5.2]. That is to say that the tangent bundle functor $U : \mathbb{T}\mathrm{opos}_\infty \to \mathbb{T}\mathrm{opos}_\infty$ is given by the exponential objects

$$U(\mathcal{X}) = \mathcal{X}^{\mathcal{T}}$$

for some object $\mathcal{T}$ in the ∞-category $\mathbb{T}\mathrm{opos}_\infty$, with the tangent structure on U arising from so-called *infinitesimal* structure on $\mathcal{T}$. This picture follows the same pattern as the tangent category associated to a model of Synthetic Differential Geometry (SDG) [CC14, 5.10], and we wonder which other features of SDG have a counterpart in the tangent ∞-category $\mathbb{T}\mathrm{opos}_\infty$. That question is not explored here.

The infinitesimal object $\mathcal{T}$ is the ∞-topos of 'parameterized spectra', also known as the Goodwillie tangent bundle $T(\mathcal{S})$ on the ∞-category of spaces $\mathcal{S}$, with infinitesimal structure determined by the Goodwillie tangent structure. The exponential objects $U(\mathcal{X}) = \mathcal{X}^{T(\mathcal{S})}$, however, do not seem to have a simple description, and we do not have a good understanding of the geometric tangent structure in its entirety.

We do offer a perspective on the geometric tangent structure by looking at its restriction to the subcategory of $\mathbb{T}\mathrm{opos}_\infty$ consisting of those ∞-toposes that are *injective* in a sense that generalizes Johnstone's notion of injective 1-topos [Joh81]. We prove that the geometric tangent structure on injective ∞-toposes is equivalent, via the '∞-category of points' functor, to the Goodwillie tangent structure (restricted to those ∞-categories that are presentable and compactly-assembled [Lur18, 21.1.2]). We therefore view the geometric tangent structure as an *extension* of the Goodwillie structure, to non-injective ∞-toposes, in addition to being its dual.

Here is an outline of the paper. In Section 1 we introduce the notion of 'infinitesimal object' in an ∞-category $\mathbb{X}$ with finite products (or, more generally, in any monoidal ∞-category), and we describe what it means for a tangent structure to be represented, or corepresented, by an infinitesimal object. These definitions extend to ∞-categories the corresponding notions for tangent categories due to Cockett and Cruttwell [CC14, Sec. 5].

In Section 2 we turn to ∞-toposes, and introduce the two perspectives given by the ∞-categories $\mathbb{T}\mathrm{opos}_\infty$ and $\mathbb{L}\mathrm{ogos}_\infty = \mathbb{T}\mathrm{opos}_\infty^{op}$. We then construct an infinitesimal object $\mathcal{T}$ in $\mathbb{T}\mathrm{opos}_\infty$ whose underlying ∞-topos is $T(\mathcal{S})$, and we show that $\mathcal{T}$ represents and corepresents tangent structures on $\mathbb{T}\mathrm{opos}_\infty$ and $\mathbb{L}\mathrm{ogos}_\infty$ respectively. As mentioned, we prove that the latter of these is the restriction of the Goodwillie tangent structure of [BBC21] to the ∞-toposes.

Finally in Section 3 we give our description (Theorem 3.6) of the geometric tangent structure on the subcategory of $\mathbb{T}\mathrm{opos}_\infty$ consisting of the injective ∞-toposes, and make some explicit calculations that arise from that result.

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1. Representable tangent ∞-categories

The goal of this section is to extend Cockett and Cruttwell's notion of *representable* tangent category [CC14, Sec. 5.2] to the tangent ∞-categories of [BBC21, 2.1]. We start by developing the notion of *infinitesimal object* in the sense of [CC14, 5.6]. As with extending other tangent category notions to ∞-categories, we do this by re-expressing the Cockett-Cruttwell definition in terms of the category of Weil-algebras that Leung used in [Leu17] to characterize tangent structures.

Throughout this paper we rely on notation and definitions from [BBC21]. In particular, let $\mathbb{W}\mathrm{eil}$ be the monoidal category of *Weil-algebras* of [BBC21, 1.1]. The objects of $\mathbb{W}\mathrm{eil}$ are the augmented commutative semi-rings of the form

$$\mathbb{N}[x_1, \dots, x_n]/(x_i x_j \mid i \sim j)$$

where the relations are quadratic monomials determined by equivalence relations on the sets $\{1, \dots, n\}$, for $n \geq 0$. Morphisms in $\mathbb{W}\mathrm{eil}$ are semi-ring homomorphisms that commute with the augmentations, and monoidal structure is given by the tensor product which is also the coproduct. Certain pullback squares in the category $\mathbb{W}\mathrm{eil}$ play a crucial role in the definition of tangent structure. We refer to those diagrams as the *tangent pullbacks*. Their precise definition is not important in this paper; for full details see [BBC21, 1.10, 1.12].

We now give our notion of infinitesimal object in a monoidal ∞-category.

Definition 1.1. Let $\mathbb{X}^{\boxtimes}$ be a monoidal ∞-category, and let $\mathbb{X}^{op,\boxtimes}$ denote the corresponding monoidal structure on the opposite ∞-category $\mathbb{X}^{op}$. An *infinitesimal object* in $\mathbb{X}^{\boxtimes}$ is a monoidal functor

$$\mathcal{D}^{\bullet} : \mathbb{W}\mathrm{eil}^{\otimes} \to \mathbb{X}^{op,\boxtimes}.$$

for which the underlying functor $\mathbb{W}\mathrm{eil} \to \mathbb{X}^{op}$ preserves the tangent pullbacks (i.e. maps tangent pullbacks in $\mathbb{W}\mathrm{eil}$ to pushouts in $\mathbb{X}$). These monoidal functors (and their monoidal natural transformations) form an ∞-category (see [Lur17, 2.1.3.7]), whose opposite[1] we refer to as the *∞-category of infinitesimal objects in* $\mathbb{X}^{\boxtimes}$, denoted $\mathbb{I}\mathrm{nf}(\mathbb{X}^{\boxtimes})$.

Example 1.2. If $\mathbb{X}$ is any ∞-category with finite products, then we refer to an infinitesimal object in the cartesian monoidal ∞-category $\mathbb{X}^{\times}$ simply as an *infinitesimal object in* $\mathbb{X}$. We write $\mathbb{I}\mathrm{nf}(\mathbb{X})$ for the ∞-category of these infinitesimal objects.

Example 1.3. Let $\mathbb{X}$ be an ordinary category with finite products. Our notion of infinitesimal object in $\mathbb{X}$ agrees with that given by Cockett-Cruttwell [CC14, 5.6] except that we do *not* require that the objects $\mathcal{D}^{A}$ are exponentiable (the axiom there labelled [**Infsml.6**]). That condition is added in Proposition 1.9 below to explain when $\mathcal{D}^{\bullet}$ represents a tangent structure on $\mathbb{X}$.

Remark 1.4. In the language of [BBC21, 5.7], an infinitesimal object in a monoidal ∞-category $\mathbb{X}^{\boxtimes}$ is precisely a tangent object in the $(\infty, 2)$-category $\underline{\mathbb{X}}^{op,\boxtimes}$ that has a single object, mapping ∞-category $\mathbb{X}^{op}$, and composition given by the monoidal structure $\boxtimes$.

In the case that the monoidal structure on $\mathbb{X}$ is given by the cartesian product, as in Example 1.2, there is a particularly simple way to identify infinitesimal objects.

Proposition 1.5. *Let $\mathbb{X}$ be an ∞-category with finite products. Taking underlying functors determines an equivalence between $\mathbb{I}\mathrm{nf}(\mathbb{X})$ and the full subcategory of* $\mathrm{Fun}(\mathbb{W}\mathrm{eil}^{op}, \mathbb{X})$ *whose objects are the functors* $\mathcal{D}^{\bullet} : \mathbb{W}\mathrm{eil}^{op} \to \mathbb{X}$ *such that*

[1] We take opposites here to ensure that morphisms in $\mathbb{I}\mathrm{nf}(\mathbb{X}^{\otimes})$ are built from morphisms in $\mathbb{X}$ rather than $\mathbb{X}^{op}$.

(1) $\mathcal{D}^{\mathbb{N}}$ *is a terminal object in* $\mathbb{X}$*;*
(2) *for Weil-algebras* A, A' *there is an equivalence in* $\mathbb{X}$

$$\mathcal{D}^{A \otimes A'} \xrightarrow{\sim} \mathcal{D}^{A} \times \mathcal{D}^{A'},$$

induced by the canonical maps from A *and* A' *to their coproduct* $A \otimes A'$*;*
(3) $\mathcal{D}^{\bullet}$ *maps the tangent pullbacks in* $\mathbb{W}\mathrm{eil}$ *to pushouts in* $\mathbb{X}$*.*

Proof. Since the monoidal structures on both $\mathbb{W}\mathrm{eil}$ and $\mathbb{X}^{op}$ are given by the coproduct, we can apply [Lur17, 2.4.3.8] to identify the ∞-category of *lax* monoidal functors $\mathbb{W}\mathrm{eil}^{\otimes} \to \mathbb{X}^{op,\times}$ with the ∞-category of *all* functors $\mathcal{D}^{\bullet} : \mathbb{W}\mathrm{eil} \to \mathbb{X}^{op}$. Conditions (1) and (2) on $\mathcal{D}^{\bullet}$ correspond to the case where that lax monoidal functor is monoidal, and (3) is the condition that this monoidal functor is an infinitesimal structure. □

We now consider two ways in which an infinitesimal object can determine a tangent structure, corresponding to those described in [CC14, Prop. 5.7 and Cor. 5.18]. First we recall the definition of tangent ∞-category from [BBC21, 2.1].

Definition 1.6. A *tangent structure* on an ∞-category $\mathbb{Y}$ is a monoidal functor

$$T : \mathbb{W}\mathrm{eil}^{\otimes} \to \mathrm{End}(\mathbb{Y})^{\circ}$$

for which the underlying functor $\mathbb{W}\mathrm{eil} \to \mathrm{End}(\mathbb{Y})$ preserves the tangent pullbacks. Here $\mathrm{End}(\mathbb{Y})^{\circ}$ is the ∞-category of endofunctors on $\mathbb{Y}$ with monoidal structure given by composition.

Proposition 1.7. *Let* $\mathbb{X}^{\boxtimes}$ *be a monoidal* ∞*-category for which the monoidal structure* $\boxtimes$ *commutes with pushouts in its first variable, and let* $\mathcal{D}^{\bullet}$ *be an infinitesimal object in* $\mathbb{X}^{\boxtimes}$*. Then there is a tangent structure on the* ∞*-category* $\mathbb{X}^{op}$ *given by the* $\mathbb{W}\mathrm{eil}$*-action map*

$$T : \mathbb{W}\mathrm{eil} \times \mathbb{X}^{op} \to \mathbb{X}^{op}; \quad T^{A}(\mathcal{C}) = \mathcal{D}^{A} \boxtimes \mathcal{C}$$

for a Weil-algebra A *and* $\mathcal{C} \in \mathbb{X}$*.*

Proof. The monoidal structure on $\mathbb{X}$ determines a monoidal functor

$$\boxtimes : \mathbb{X}^{op,\boxtimes} \to \mathrm{End}(\mathbb{X}^{op})^{\circ}; \quad \mathcal{D} \mapsto \mathcal{D} \boxtimes -$$

which, by hypothesis, preserves pullbacks. Composing the infinitesimal object $\mathcal{D}^{\bullet} : \mathbb{W}\mathrm{eil}^{\otimes} \to \mathbb{X}^{op,\boxtimes}$ with the monoidal functor $\boxtimes$, we get a monoidal functor

$$\mathcal{D}^{\bullet} \boxtimes - : \mathbb{W}\mathrm{eil}^{\otimes} \to \mathrm{End}(\mathbb{X}^{op})^{\circ}$$

which preserves the tangent pullbacks. Thus T is a tangent structure on $\mathbb{X}^{op}$; see [BBC21, 2.1]. □

Definition 1.8. We say the tangent structure T in Proposition 1.7 is *corepresented* by the infinitesimal object $\mathcal{D}^\bullet$. A tangent structure is *corepresentable* if it is equivalent to a tangent structure corepresented by some infinitesimal object.

Proposition 1.9. *Let $\mathbb{X}^{\boxtimes}$ be a monoidal ∞-category such that the monoidal product $\boxtimes$ preserves pushouts in its first variable. Let $\mathcal{D}^\bullet$ be an infinitesimal object in $\mathbb{X}^{\boxtimes}$ such that for each Weil-algebra A, the functor $\mathcal{D}^A \boxtimes - : \mathbb{X} \to \mathbb{X}$ admits a right adjoint $\mathrm{Map}^{\boxtimes}_{\mathbb{X}}(\mathcal{D}^A, -)$. Then there is a tangent structure on the ∞-category $\mathbb{X}$ given by*

$$U : \mathbb{W}\mathrm{eil} \times \mathbb{X} \to \mathbb{X}; \quad U^A(\mathcal{C}) = \mathrm{Map}^{\boxtimes}_{\mathbb{X}}(\mathcal{D}^A, \mathcal{C}).$$

Proof. Note that we do not assume that the monoidal structure $\boxtimes$ as a whole is *closed*, only that certain specific functors admit a right adjoint. Our first task is to show that those right adjoints can be chosen functorially.

Let $\mathrm{End}^L(\mathbb{X})^\circ \subseteq \mathrm{End}(\mathbb{X})^\circ$ be the full (monoidal) subcategory whose objects are the left adjoint functors $\mathbb{X} \to \mathbb{X}$, and similarly for $\mathrm{End}^R(\mathbb{X})$. Then there is an equivalence of monoidal ∞-categories

$$\mathrm{adj} : \mathrm{End}^L(\mathbb{X})^\circ \xrightarrow{\sim} \mathrm{End}^R(\mathbb{X})^\circ$$

that sends a functor to some choice of its right adjoint. Such an equivalence can be constructed in a similar manner to that in [Lur09, 5.5.3.4].

By hypothesis, the composite

$$\mathbb{W}\mathrm{eil}^{\otimes} \xrightarrow{\mathcal{D}^\bullet} \mathbb{X}^{op,\boxtimes} \xrightarrow{\boxtimes} \mathrm{End}(\mathbb{X}^{op})^\circ \xrightarrow[op]{\sim} \mathrm{End}(\mathbb{X})^\circ$$

takes values in $\mathrm{End}^L(\mathbb{X})^\circ$. Therefore we can form the composite monoidal functor

$$\mathbb{W}\mathrm{eil}^{\otimes} \to \mathrm{End}^L(\mathbb{X})^\circ \xrightarrow[\mathrm{adj}]{\sim} \mathrm{End}^R(\mathbb{X})^\circ \to \mathrm{End}(\mathbb{X})^\circ.$$

It remains to show that the underlying functor $A \mapsto \mathrm{Map}^{\boxtimes}_{\mathbb{X}}(\mathcal{D}^A, -)$ preserves each of the tangent pullbacks in $\mathbb{W}\mathrm{eil}$. Suppose

$$\begin{array}{ccc} A & \longrightarrow & A_1 \\ \downarrow & & \downarrow \\ A_2 & \longrightarrow & A_0 \end{array}$$

is one of those tangent pullbacks. Then it is sufficient to show that for all $\mathcal{C}, \mathcal{E} \in \mathbb{X}$, the following diagram is a pullback of mapping spaces:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbb{X}}(\mathcal{E}, \mathrm{Map}^{\boxtimes}_{\mathbb{X}}(\mathcal{D}^{A}, \mathcal{C})) & \longrightarrow & \mathrm{Hom}_{\mathbb{X}}(\mathcal{E}, \mathrm{Map}^{\boxtimes}_{\mathbb{X}}(\mathcal{D}^{A_1}, \mathcal{C})) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathbb{X}}(\mathcal{E}, \mathrm{Map}^{\boxtimes}_{\mathbb{X}}(\mathcal{D}^{A_2}, \mathcal{C})) & \longrightarrow & \mathrm{Hom}_{\mathbb{X}}(\mathcal{E}, \mathrm{Map}^{\boxtimes}_{\mathbb{X}}(\mathcal{D}^{A_0}, \mathcal{C}))
\end{array}$$

We can write this diagram equivalently as

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbb{X}}(\mathcal{D}^{A} \boxtimes \mathcal{E}, \mathcal{C}) & \longrightarrow & \mathrm{Hom}_{\mathbb{X}}(\mathcal{D}^{A_1} \boxtimes \mathcal{E}, \mathcal{C}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathbb{X}}(\mathcal{D}^{A_2} \boxtimes \mathcal{E}, \mathcal{C}) & \longrightarrow & \mathrm{Hom}_{\mathbb{X}}(\mathcal{D}^{A_0} \boxtimes \mathcal{E}, \mathcal{C})
\end{array}$$

which is a pullback since $- \boxtimes \mathcal{E}$ preserves pushouts by hypothesis. □

Definition 1.10. We say that the tangent structure U in Proposition 1.9 is *represented* by the infinitesimal object $\mathcal{D}^{\bullet}$. A tangent structure is *representable* if it is equivalent to one represented by some infinitesimal object.

Definition 1.11. Tangent structures on $\mathbb{X}$ and $\mathbb{X}^{op}$ are *dual* if they are, respectively, represented and corepresented by the same infinitesimal object. It follows from comparing the hypotheses in Propositions 1.7 and 1.9 that any representable tangent structure on $\mathbb{X}$ has a dual tangent structure on $\mathbb{X}^{op}$.

2. ∞-TOPOSES

The main purpose of this paper is to construct dual tangent structures on a certain ∞-category $\mathbb{T}\mathrm{opos}_{\infty}$ and its opposite, whose objects are ∞-toposes. In this section we introduce that ∞-category and construct the infinitesimal object $\mathcal{T}^{\bullet}$ that (co)represents those tangent structures. Our main reference for the ∞-category $\mathbb{T}\mathrm{opos}_{\infty}$ is [Lur09, Sec. 6.3] where it is denoted $\mathcal{R}\mathcal{T}\mathrm{op}_{\infty}$.

To define the ∞-category $\mathbb{T}\mathrm{opos}_{\infty}$ we have to pay some attention to size issues. We assume three nested Grothendieck universes and refer to simplicial sets of these sizes as *small*, *large* and *very large*, respectively. Let $\mathbb{C}\mathrm{at}_{\infty}$ denote the (very large) ∞-category of large ∞-categories. One of the objects in $\mathbb{C}\mathrm{at}_{\infty}$ is the (large) ∞-category $\mathcal{S}$ of small spaces, i.e. Kan complexes.

Definition 2.1. An *∞-topos* is a (large) ∞-category $\mathcal{X}$ that is an accessible left exact localization of the ∞-category $\mathcal{P}(\mathsf{C}) = \mathrm{Fun}(\mathsf{C}^{op}, \mathcal{S})$ of presheaves on some small ∞-category C. In other words, there is some small ∞-category C and an adjunction

$$\mathcal{P}(\mathsf{C}) \underset{g}{\overset{f}{\underset{\perp}{\rightleftarrows}}} \mathcal{X} \tag{2.2}$$

such that g is fully faithful and accessible (preserves κ-filtered colimits for some small regular cardinal κ), and f preserves finite limits.

Example 2.3. The presheaf ∞-category $\mathcal{P}(\mathsf{C})$, for a small ∞-category C, is an ∞-topos. In particular $\mathcal{S}$ is an ∞-topos which, by [Lur09, 6.3.4.1], is a terminal object in the ∞-category $\mathbb{T}\mathrm{opos}_\infty$ which we now introduce.

Definition 2.4. Let $\mathbb{T}\mathrm{opos}_\infty$ denote the subcategory of $\mathbb{C}\mathrm{at}_\infty$ whose objects are the ∞-toposes and whose morphism are the *geometric morphisms*, i.e. those functors $F : \mathcal{X} \to \mathcal{Y}$ which admit a left adjoint $\mathcal{Y} \to \mathcal{X}$ that preserves finite limits.

The opposite ∞-category $\mathbb{T}\mathrm{opos}_\infty^{op}$ is also equivalent to a subcategory of $\mathbb{C}\mathrm{at}_\infty$ which, following Anel and Joyal [AJ19], we denote by $\mathbb{L}\mathrm{ogos}_\infty$.

Definition 2.5. Let $\mathbb{L}\mathrm{ogos}_\infty$ be the subcategory of $\mathbb{C}\mathrm{at}_\infty$ whose objects are the ∞-toposes and morphisms are the functors that preserve small colimits and finite limits. There is an equivalence $\mathbb{L}\mathrm{ogos}_\infty \simeq \mathbb{T}\mathrm{opos}_\infty^{op}$ that is the identity on objects and maps a functor $\mathcal{Y} \to \mathcal{X}$ in $\mathbb{L}\mathrm{ogos}_\infty$ to the right adjoint guaranteed by the Adjoint Functor Theorem [Lur09, 5.5.2.9], a geometric morphism $\mathcal{X} \to \mathcal{Y}$; see [Lur09, 6.3.1.8].

The remainder of this section is devoted to the construction of an infinitesimal object in $\mathbb{T}\mathrm{opos}_\infty \simeq \mathbb{L}\mathrm{ogos}_\infty^{op}$, i.e. a monoidal functor

$$\mathcal{T}^\bullet : \mathbb{W}\mathrm{eil}^\otimes \to \mathbb{L}\mathrm{ogos}_\infty^\boxtimes$$

where $\boxtimes$ denotes the coproduct in the ∞-category $\mathbb{L}\mathrm{ogos}_\infty$, or equivalently the product in $\mathbb{T}\mathrm{opos}_\infty$.

The infinitesimal object $\mathcal{T}^\bullet$ is derived from the Goodwillie tangent structure constructed in [BBC21] which we now recall. See [BBC21, 2.1] for the notion of tangent structure on an ∞-category, and see [BBC21, Sec. 7] for the basic facets of the Goodwillie tangent structure.

Definition 2.6. Let $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ be the ∞-category of (large) differentiable[2] ∞-categories and sequential-colimit-preserving functors. Let $\mathsf{S}_{\mathsf{fin},*}$ denote the (small) ∞-category of pointed finite spaces.

[2] An ∞-category is *differentiable* if it has finite limits and sequential colimits which commute. Any ∞-topos is differentiable by [Lur09, 7.3.4.7].

The *Goodwillie tangent structure* on $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ is a map

$$T : \mathbb{W}\mathrm{eil} \times \mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}} \to \mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$$

given on a Weil-algebra A with n generators, and differentiable ∞-category $\mathcal{C}$, by the subcategory

$$T^A(\mathcal{C}) = \mathrm{Exc}^A(\mathsf{S}^n_{\mathsf{fin},*}, \mathcal{C}) \subseteq \mathrm{Fun}(\mathsf{S}^n_{\mathsf{fin},*}, \mathcal{C}) \tag{2.7}$$

of functors $\mathsf{S}^n_{\mathsf{fin},*} \to \mathcal{C}$ that are A-*excisive*[3] in the sense described in [BBC21, 7.1]. By [BBC21, 7.5], the inclusion (2.7) admits a left adjoint P_A which preserves finite limits.

The action of T on morphisms (in $\mathbb{W}\mathrm{eil}$ and $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$) is described in detail in [BBC21, 7.7 and 7.14]. For a Weil-algebra morphism $\phi : A \to A'$ and (sequential-colimit-preserving) functor $G : \mathcal{C} \to \mathcal{D}$, we have

$$T^\phi(F) : T^A(\mathcal{C}) \to T^{A'}(\mathcal{D}); \quad L \mapsto P_{A'}(GL\tilde{\phi})$$

where $\tilde{\phi} : \mathsf{S}^{n'}_{\mathsf{fin},*} \to \mathsf{S}^n_{\mathsf{fin},*}$ is a functor built to the same pattern as the algebra homomorphism ϕ; see [BBC21, 7.12].

Proposition 2.8. *The Goodwillie tangent structure on* $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ *restricts to a tangent structure on the subcategory* $\mathbb{L}\mathrm{ogos}_\infty \subseteq \mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$.

Proof. Suppose first that $\mathcal{X}$ is an ∞-topos and A is a Weil-algebra with n generators. By [BBC21, 7.5], the ∞-category $T^A(\mathcal{X})$ is an accessible left exact localization of the ∞-topos $\mathrm{Fun}(\mathsf{S}^n_{\mathsf{fin},*}, \mathcal{X})$, hence $T^A(\mathcal{X})$ is an ∞-topos.

Now let $G : \mathcal{X} \to \mathcal{Y}$ be a morphism in $\mathbb{L}\mathrm{ogos}_\infty$ and $\phi : A \to A'$ a Weil-algebra morphism. We have to show that the functor

$$T^\phi(G) : T^A(\mathcal{X}) \to T^{A'}(\mathcal{Y}); \quad L \mapsto P_{A'}(GL\tilde{\phi})$$

is also in $\mathbb{L}\mathrm{ogos}_\infty$. Finite limits in $T^A(\mathcal{X})$ and $T^{A'}(\mathcal{Y})$ are calculated objectwise, and both G and $P_{A'}$ preserve those finite limits, so $T^\phi(G)$ preserves finite limits.

Let (L_α) be a diagram in $T^A(\mathcal{X})$ with colimit L. Then we have an equivalence $L \simeq P_A(\mathrm{colim}\, L_\alpha)$ where colim denotes the (objectwise) colimit calculated in $\mathrm{Fun}(\mathsf{S}^n_{\mathsf{fin},*}, \mathcal{C})$. Then there is a sequence of equivalences:

$$\begin{aligned} P_{A'}(GL\tilde{\phi}) &\simeq P_{A'}(GP_A(\mathrm{colim}\, L_\alpha)\tilde{\phi}) \\ &\simeq P_{A'}(G(\mathrm{colim}\, L_\alpha)\tilde{\phi}) \\ &\simeq P_{A'}(\mathrm{colim}\, GL_\alpha\tilde{\phi}) \end{aligned}$$

[3] A functor is *excisive* if it maps pushout squares to pullbacks. The notion of A-excisive is a multivariable generalization of excisive that reflects the structure of the Weil-algebra A.

where we have used equivalences of the form [BBC21, 7.11 and 7.25] to identify the first and second lines, and the fact that G preserves colimits to identify the second and third.

Therefore $T^{\phi}(G)$ preserves colimits, which completes the proof that $T^{\phi}(G)$ is a morphism in $\mathbb{L}\mathrm{ogos}_{\infty}$, and hence that the $\mathbb{W}\mathrm{eil}$-action on $\mathbb{C}\mathrm{at}_{\infty}^{\mathrm{diff}}$ restricts to a functor

$$T : \mathbb{W}\mathrm{eil} \times \mathbb{L}\mathrm{ogos}_{\infty} \to \mathbb{L}\mathrm{ogos}_{\infty}.$$

It remains to show that T preserves the tangent pullbacks in $\mathbb{W}\mathrm{eil}$. Since limits in $\mathbb{L}\mathrm{ogos}_{\infty}$ are calculated in $\mathbb{C}\mathrm{at}_{\infty}$ by [Lur09, 6.3.2.3], that claim follows from [BBC21, 7.36 and 7.38]. $\square$

We now construct the desired infinitesimal object.

Definition 2.9. Define a functor

$$\mathfrak{T}^{\bullet} : \mathbb{W}\mathrm{eil} \to \mathbb{L}\mathrm{ogos}_{\infty}$$

by

$$\mathfrak{T}^{A} := T^{A}(\mathcal{S}),$$

i.e. by evaluating the Goodwillie tangent structure at the ∞-topos $\mathcal{S}$ of spaces.

Proposition 2.10. *The functor $\mathfrak{T}^{\bullet}$ is the underlying functor of an infinitesimal object in $\mathbb{T}\mathrm{opos}_{\infty} \simeq \mathbb{L}\mathrm{ogos}_{\infty}^{op}$.*

Proof. We apply Proposition 1.5:

(1) By [Lur09, 6.3.4.1], $\mathfrak{T}^{\mathbb{N}} = \mathcal{S}$ is the terminal object in $\mathbb{T}\mathrm{opos}_{\infty}$.

(2) The canonical map

$$T^{A \otimes A'}(\mathcal{S}) \to T^{A}(\mathcal{S}) \boxtimes T^{A'}(\mathcal{S})$$

is an equivalence of ∞-toposes; this claim follows from Lemma 2.15 below by taking $\mathcal{X} = T^{A'}(\mathcal{S})$ and noting that $T^{A \otimes A'}(\mathcal{S}) = T^{A}(T^{A'}(\mathcal{S}))$.

(3) For each tangent pullback in $\mathbb{W}\mathrm{eil}$, the corresponding diagram

$$\begin{tikzcd} T^{A}(\mathcal{S}) \arrow[r] \arrow[d] & T^{A_1}(\mathcal{S}) \arrow[d] \\ T^{A_2}(\mathcal{S}) \arrow[r] & T^{A_0}(\mathcal{S}) \end{tikzcd}$$

is a pullback in $\mathbb{L}\mathrm{ogos}_{\infty}$ (and hence a pushout in $\mathbb{T}\mathrm{opos}_{\infty}$). This claim is part of the condition that T is a tangent structure on $\mathbb{L}\mathrm{ogos}_{\infty}$, as verified in the last part of the proof of Proposition 2.8.

□

To show that the infinitesimal object $\mathcal{T}^\bullet$ represents, and corepresents, tangent structures on the ∞-categories $\mathbb{T}\mathrm{opos}_\infty$ and $\mathbb{L}\mathrm{ogos}_\infty$ respectively, we verify the conditions of Propositions 1.7 and 1.9.

Proposition 2.11. *The product $\boxtimes$ on $\mathbb{T}\mathrm{opos}_\infty$ preserves pushouts in each variable individually.*

Proof. We use the fact, e.g. see [AL18, 2.15], that the coproduct $\boxtimes$ in $\mathbb{L}\mathrm{ogos}_\infty$ is given by the tensor product of cocomplete ∞-categories; see [Lur17, 4.8.1]. Then [AL18, 4.24] tells us that for ∞-toposes $\mathcal{Y}, \mathcal{X}$, we have

$$\mathcal{Y} \boxtimes \mathcal{X} \simeq \mathrm{Fun}^{\lim}(\mathcal{X}^{op}, \mathcal{Y})$$

where the right-hand side is the ∞-category functors $\mathcal{Y}^{op} \to \mathcal{X}$ that preserve small limits.

We therefore have to show that $\mathrm{Fun}^{\lim}(\mathcal{X}^{op}, -)$ preserves pushouts in $\mathbb{T}\mathrm{opos}_\infty$ which, by [Lur09, 6.3.2.3], are pullbacks in $\mathbb{C}\mathrm{at}_\infty$. That claim is a consequence of [RV20, 6.4.12] which implies that pullbacks in the ∞-cosmos of ∞-categories with small limits are given by pullbacks in the ∞-cosmos of all ∞-categories. □

Proposition 2.12. *For each Weil-algebra A, the functor*

$$\mathcal{T}^A \boxtimes - : \mathbb{T}\mathrm{opos}_\infty \to \mathbb{T}\mathrm{opos}_\infty$$

admits a right adjoint.

Proof. Anel and Lejay [AL18, 4.37] show that any compactly-generated ∞-topos is exponentiable, so it is sufficient to show that $\mathcal{T}^A = \mathrm{Exc}^A(\mathsf{S}^n_{\mathsf{fin},*}, \mathcal{S})$ is compactly-generated. It follows from [Lur09, 5.3.5.12] that the presheaf ∞-category $\mathrm{Fun}(\mathsf{S}^n_{\mathsf{fin},*}, \mathcal{S})$ is compactly-generated, so by [Lur09, 5.5.7.3] it is sufficient to note that $\mathrm{Exc}^A(\mathsf{S}_{\mathsf{fin},*}, \mathcal{S})$ is closed under filtered colimits in $\mathrm{Fun}(\mathsf{S}^n_{\mathsf{fin},*}, \mathcal{S})$, which follows from the fact that filtered colimits in $\mathcal{S}$ commute with pullbacks. □

Theorem 2.13. *The infinitesimal object $\mathcal{T}^\bullet$ represents a tangent structure U on the ∞-category $\mathbb{T}\mathrm{opos}_\infty$ and corepresents a tangent structure T on the ∞-category $\mathbb{L}\mathrm{ogos}_\infty$.*

Proof. We apply Propositions 1.9 and 1.7, respectively, using the results of Propositions 2.11 and 2.12. □

We refer to the tangent structure U on $\mathbb{T}\mathrm{opos}_\infty$ as the *geometric tangent structure*, and we begin the study of that structure in the next section. The corepresented tangent structure T turns out to be much more familiar.

Proposition 2.14. *The corepresentable tangent structure of Theorem 2.13 is equivalent to the restriction of the Goodwillie tangent structure of* [BBC21] *to the subcategory* $\mathbb{L}\mathrm{ogos}_\infty \subseteq \mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$*, as described in Proposition 2.8.*

Proof. We have to show that the following diagram of monoidal functors commutes up to monoidal equivalence.

$$\begin{tikzcd}
\mathbb{W}\mathrm{eil}^{\otimes} \arrow[r, "\mathcal{T}^{\bullet}"] \arrow[dr, "T"'] & \mathbb{L}\mathrm{ogos}_\infty^{\boxtimes} \arrow[d, "\boxtimes"] \\
 & \mathrm{End}(\mathbb{L}\mathrm{ogos}_\infty)^{\circ}
\end{tikzcd}$$

This claim is a consequence of the following lemma. $\square$

Lemma 2.15. *Let $\mathcal{X}$ be an ∞-topos, and A a Weil-algebra. Then there is a canonical equivalence in* $\mathbb{L}\mathrm{ogos}_\infty$

$$T^A(\mathcal{S}) \boxtimes \mathcal{X} \xrightarrow{\sim} T^A(\mathcal{X})$$

built from the maps $T^A(!) : T^A(\mathcal{S}) \to T^A(\mathcal{X})$ and $T^\eta(\mathcal{X}) : \mathcal{X} \to T^A(\mathcal{X})$, where $! : \mathcal{S} \to \mathcal{X}$ and $\eta : \mathbb{N} \to A$ are the maps from the initial objects in $\mathbb{L}\mathrm{ogos}_\infty$ *and* $\mathbb{W}\mathrm{eil}$ *respectively.*

Proof. Using the approach from the proof of Proposition 2.11, it is sufficient to show that the canonical map

$$T^A(\mathcal{X}) \to \mathrm{Fun}^{\lim}(\mathcal{X}^{op}, T^A(\mathcal{S})); \quad L \mapsto \mathrm{Hom}_{\mathcal{X}}(-, L)$$

is an equivalence. Noting that limits in $T^A(\mathcal{S}) = \mathrm{Exc}^A(\mathsf{S}^n_{\mathsf{fin},*}, \mathcal{S})$ are calculated objectwise, we can rewrite the target of that map as the ∞-category of A-excisive functors $\mathsf{S}^n_{\mathsf{fin},*} \to \mathrm{Fun}^{\lim}(\mathcal{X}^{op}, \mathcal{S})$, since pullbacks in $\mathrm{Fun}^{\lim}(\mathcal{X}^{op}, \mathcal{S})$ are also calculated objectwise. Our claim then follows by noting that the map

$$\mathcal{X} \to \mathrm{Fun}^{\lim}(\mathcal{X}^{op}, \mathcal{S}); \quad \mathrm{Hom}_{\mathcal{X}}(-, L)$$

is an equivalence by [AL18, 4.24] again. $\square$

3. The geometric tangent structure

The aim of this section is to begin a study of the geometric tangent structure on $\mathbb{T}\mathrm{opos}_\infty$ given by Thorem 2.13. By definition, the tangent bundle construction for this tangent structure,

$$U(\mathcal{X}) = \mathcal{X}^{T(\mathcal{S})},$$

is the exponential object for the ∞-topos $T(\mathcal{S})$. These exponential objects do not appear to be easy to calculate, though a general construction can be gleaned from the proofs of [AL18, 4.33] or [Lur18, 21.1.6.12].

Those approaches proceed by first calculating the exponential object for a collection of ∞-toposes that are injective in the sense of Definition 3.1 below. Since any ∞-topos $\mathcal{X}$ can be written as the pullback of a diagram of injective ∞-toposes [Lur18, 21.1.6.16], and since U preserves pullbacks, one might be able to recover an explicit description of $U(\mathcal{X})$ from those calculations, though we do not attempt that here.

It turns out that the geometric tangent structure for injective ∞-toposes has a compelling description. We prove in Theorem 3.6 that the '∞-category of points' construction determines an equivalence between that geometric tangent structure and the Goodwillie tangent structure restricted to the presentable compactly-assembled ∞-categories of [Lur18, 21.1.2]. Thus, on injective ∞-toposes at least, one can view the geometric tangent structure as simply a different incarnation of the Goodwillie structure.

Definition 3.1. An ∞-topos $\mathcal{X}$ is *injective* if $\mathcal{X}$ is a retract, in $\mathbb{T}\mathrm{opos}_\infty$, of a presheaf ∞-category $\mathcal{P}(\mathsf{D})$ where D is a small ∞-category that has finite limits. Let $\mathbb{I}\mathrm{nj}\mathbb{T}\mathrm{opos}_\infty$ be the full subcategory of $\mathbb{T}\mathrm{opos}_\infty$ consisting of the injective ∞-toposes.

Remark 3.2. An ∞-topos is injective if and only if it satisfies the equivalent conditions of [Lur18, 21.1.5.4]; our definition is an intermediate step in proving (4) implies (1) in that result. Our definition is also equivalent to that of Anel and Lejay in [AL18, 4.6]; combine (4) of [Lur18, 21.1.5.4] with [AL18, 2.6].

The attraction of injective ∞-toposes is that they can be recovered from their ∞-categories of 'points'.

Definition 3.3. The *∞-category of points* of an ∞-topos $\mathcal{X}$ is the ∞-category

$$p(\mathcal{X}) := \mathrm{Fun}^*(\mathcal{X}, \mathcal{S})$$

of functors $\mathcal{X} \to \mathcal{S}$ that preserve small colimits and finite limits, i.e. the geometric morphisms $\mathcal{S} \to \mathcal{X}$. Since $\mathcal{S}$ is the terminal object in $\mathbb{T}\mathrm{opos}_\infty$, the objects of $p(\mathcal{X})$ are indeed the 'generalized points' of the ∞-topos $\mathcal{X}$. By [Lur18, 21.1.1.6], the construction of $p(\mathcal{X})$ extends to a functor

$$p : \mathbb{T}\mathrm{opos}_\infty \to \mathbb{C}\mathrm{at}_\infty^{\mathrm{acc},\omega}$$

whose target is the subcategory of $\mathbb{C}\mathrm{at}_\infty$ consisting of the ∞-categories that are accessible and admit filtered colimits, with morphisms the filtered-colimit-preserving functors.

Proposition 3.4. *The functor p restricts to an equivalence of ∞-categories*

$$p : \mathbb{I}\mathrm{nj}\mathbb{T}\mathrm{opos}_\infty \xrightarrow{\sim} \mathbb{C}\mathrm{at}_\infty^{\mathrm{pr,ca}} \subseteq \mathbb{C}\mathrm{at}_\infty^{\mathrm{acc},\omega}$$

whose target consists of those ∞-categories $\mathcal{C}$ that are both presentable and compactly-assembled, in the sense of [Lur18, 21.1.2.1]*. The inverse to p maps such an ∞-category $\mathcal{C}$ to the ∞-topos $\mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S})$ of filtered-colimit-preserving functors $\mathcal{C} \to \mathcal{S}$.*

Proof. The inverse map $\mathrm{Fun}^\omega(-, \mathcal{S})$ is fully faithful by [Lur18, 21.1.5.3], essentially surjective by [Lur18, 21.1.5.4(1)], and has inverse p by [Lur18, 21.1.5.1]. □

Remark 3.5. An alternative approach to the proof of Proposition 3.4 is in [AL18, 4.9] which identifies $\mathbb{C}\mathrm{at}_\infty^{\mathrm{pr,ca}}$ with the full subcategory of $\mathbb{C}\mathrm{at}_\infty^{\mathrm{acc},\omega}$ consisting of retracts of the preasheaf ∞-categories.

We now prove the main result of this section, giving a calculation of the geometric tangent structure for injective ∞-toposes.

Theorem 3.6. *The equivalence p of Proposition 3.4 underlies an equivalence of tangent structures*

$$p : (\mathbb{I}\mathrm{nj}\mathbb{T}\mathrm{opos}_\infty, U) \xrightarrow{\sim} (\mathbb{C}\mathrm{at}_\infty^{\mathrm{pr,ca}}, T)$$

between the geometric tangent structure on $\mathbb{I}\mathrm{nj}\mathbb{T}\mathrm{opos}_\infty \subseteq \mathbb{T}\mathrm{opos}_\infty$ and the Goodwillie tangent structure on $\mathbb{C}\mathrm{at}_\infty^{\mathrm{pr,ca}}$.[4]

Proof. We start by showing that the Goodwillie tangent structure restricts to $\mathbb{C}\mathrm{at}_\infty^{\mathrm{pr,ca}}$. Suppose $\mathcal{C}$ is presentable and compactly-assembled. Then, by Remark 3.5, $\mathcal{C}$ is a retract, in $\mathbb{C}\mathrm{at}_\infty^{\mathrm{acc},\omega}$, of a presheaf ∞-category. Hence $\mathrm{Fun}(\mathsf{S}^n_{\mathsf{fin},*}, \mathcal{C})$ is a retract of a presheaf ∞-category, so is also presentable and compactly-assembled. Finally, the map P_A of Definition 2.6 displays $T^A(\mathcal{C})$

[4]It is unclear to this author whether an arbitrary presentable compactly-assembled ∞-category $\mathcal{C}$ is differentiable, so $\mathbb{C}\mathrm{at}_\infty^{\mathrm{pr,ca}}$ is perhaps not a subcategory of $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$. However, the construction of the Goodwillie tangent structure in [BBC21] can be carried out with $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ replaced by the ∞-category $\mathbb{C}\mathrm{at}_\infty^{\mathrm{pr}}$ of presentable ∞-categories and filtered-colimit-preserving functors, of which $\mathbb{C}\mathrm{at}_\infty^{\mathrm{pr,ca}}$ is a full subcategory. Alternatively the reader may restrict attention to the compactly-generated ∞-categories which correspond to those ∞-toposes that are presheaves on a small ∞-category that has finite limits.

as a retract, in $\mathbb{C}\mathrm{at}_\infty^{\mathrm{acc},\omega}$, of $\mathrm{Fun}(\mathsf{S}^n_{\mathsf{fin},*}, \mathcal{C})$, so $T^A(\mathcal{C})$ is also presentable and compactly-assembled.

Now let $q : \mathbb{C}\mathrm{at}_\infty^{\mathrm{pr,ca}} \to \mathbb{I}\mathrm{nj}\mathbb{T}\mathrm{opos}_\infty$ be the inverse to p given by $q(\mathcal{C}) = \mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S})$. We then define natural equivalences

$$\alpha : qT^A \xrightarrow{\sim} U^A q$$

with components

$$\alpha_{\mathcal{C}} : \mathrm{Fun}^\omega(T^A(\mathcal{C}), \mathcal{S}) \to U^A(\mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S})) = \mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S})^{T^A(\mathcal{S})}$$

as follows.

First note that the proof of Lemma 2.15 relies purely on the identification of $\boxtimes$ with the tensor product for presentable ∞-categories, and so extends to give a canonical equivalence

$$T^A(\mathcal{S}) \boxtimes \mathcal{C} \xrightarrow{\sim} T^A(\mathcal{C})$$

for all presentable ∞-categories. By [Lur18, 21.1.4.3], and the argument of [Lur18, 21.1.6.9], we also have equivalences of ∞-toposes of the form

$$\mathrm{Fun}^\omega(\mathcal{X} \boxtimes \mathcal{C}, \mathcal{S}) \xrightarrow{\sim} \mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S})^{\mathcal{X}}.$$

Combining these two maps, with $\mathcal{X} = T^A(\mathcal{S})$, yields the desired equivalence $\alpha_{\mathcal{C}}$. Note that the existence of these equivalences also verifies that U restricts to a tangent structure on the subcategory $\mathbb{I}\mathrm{nj}\mathbb{T}\mathrm{opos}_\infty \subseteq \mathbb{T}\mathrm{opos}_\infty$.

The construction of $\alpha_{\mathcal{C}}$ is natural (in A and $\mathcal{C}$) and monoidal (with respect to the tensor product of Weil-algebras), so the maps α yield an equivalence of tangent structures with underlying functor q, whose inverse is the required tangent equivalence p. $\square$

Corollary 3.7. *For an injective ∞-topos $\mathcal{X}$:*

$$U\mathcal{X} \simeq \mathrm{Fun}^\omega(T(p\mathcal{X}), \mathcal{S}).$$

Corollary 3.8. *Let $\mathcal{X}$ be an injective ∞-topos, and let $x : \mathcal{S} \to \mathcal{X}$ be a generalized point in $\mathcal{X}$. Then the geometric tangent space $U_x\mathcal{X}$ (in $\mathbb{T}\mathrm{opos}_\infty$) exists and has ∞-category of points*

$$p(U_x\mathcal{X}) \simeq T_x(p\mathcal{X}).$$

Note, however, that we have no reason to believe that $U_x\mathcal{X}$ is injective.

Proof. By definition the tangent space is the pullback in $\mathbb{T}\mathrm{opos}_\infty$ of the form

$$\begin{tikzcd} U_x\mathcal{X} \arrow[r] \arrow[d] & U\mathcal{X} \arrow[d, "\epsilon_\mathcal{X}"] \\ \mathcal{S} \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

This pullback exists, and is preserved by each U^A, since $\mathbb{T}\mathrm{opos}_\infty$ has all limits, and U^A is a right adjoint. The functor $p : \mathbb{T}\mathrm{opos}_\infty \to \mathbb{C}\mathrm{at}_\infty^{\mathrm{acc},\omega}$ is a right adjoint by [Lur18, 21.1.1.6], so applying p we get a pullback diagram in $\mathbb{C}\mathrm{at}_\infty^{\mathrm{acc},\omega}$, and hence in $\mathbb{C}\mathrm{at}_\infty$, of the form

$$\begin{tikzcd} p(U_x\mathcal{X}) \arrow[r] \arrow[d] & T(p\mathcal{X}) \arrow[d, "\epsilon_{p\mathcal{X}}"] \\ * \arrow[r, "x"] & p\mathcal{X} \end{tikzcd}$$

which identifies $p(U_x\mathcal{X})$ with the Goodwillie tangent space $T_x(p\mathcal{X})$, as claimed. $\square$

Lurie's proof that a compactly-generated ∞-topos is exponentiable relies on a lemma [Lur18, 21.1.6.16] that says every ∞-topos is a pullback of injective ∞-toposes; see also [AL18, 2.8]. Since the geometric tangent bundle functor U is a right adjoint, we could in principle use such pullbacks, together with the calculation in Corollary 3.7, to give an explicit description of $U\mathcal{X}$ for any ∞-topos $\mathcal{X}$.

Theorem 3.6 also gives us a different perspective on the relationship between the geometric and Goodwillie tangent structures. We saw in Theorem 2.13 that those tangent structures are dual, but now we can also view the geometric tangent structure on $\mathbb{T}\mathrm{opos}_\infty$ as an *extension* of the Goodwillie tangent structure. If we think of an ∞-topos as an ∞-category (of points) together with additional information, then the geometric tangent bundle on an injective ∞-topos simply *is* the Goodwillie tangent bundle. The full geometric tangent structure on $\mathbb{T}\mathrm{opos}_\infty$ extends the Goodwillie structure to ∞-toposes for which that additional information is nontrivial.

We conclude by giving some simple calculations of the geometric tangent structure based on Corollary 3.7.

Example 3.9. For the terminal ∞-topos $\mathcal{S}$, we have

$$U(\mathcal{S}) \simeq \mathcal{S}.$$

Since U is a right adjoint and $\mathcal{S}$ is a terminal object, we did not need Theorem 3.6 to prove this fact. However, we now see that it corresponds to the calculation $T(*) \simeq *$ for the Goodwillie tangent bundle on the trivial ∞-category.

Example 3.10. Let C be a small ∞-category, and let

$$\mathcal{A}^{\mathsf{C}} := \mathcal{P}(\bar{\mathsf{C}}) = \mathrm{Fun}(\bar{\mathsf{C}}^{op}, \mathcal{S})$$

be the *affine* ∞-topos of [AL18, 2.7], where $\bar{\mathsf{C}}$ is obtained by freely adding finite limits to C. The ∞-category of points of $\mathcal{A}^{\mathsf{C}}$ is

$$p(\mathcal{A}^{\mathsf{C}}) \simeq \mathcal{P}(\mathsf{C}^{op}) = \mathrm{Fun}(\mathsf{C}, \mathcal{S})$$

and so the geometric tangent bundle is given by

$$U(\mathcal{A}^{\mathsf{C}}) \simeq \mathrm{Fun}^{\omega}(\mathrm{Fun}(\mathsf{C}, T(\mathcal{S})), \mathcal{S}).$$

By Corollary 3.8, the geometric tangent space $U_x(\mathcal{A}^{\mathsf{C}})$, for a functor $x : \mathsf{C} \to \mathcal{S}$, has ∞-category of points

$$p(U_x \mathcal{A}^{\mathsf{C}}) \simeq \mathrm{Fun}_x(\mathsf{C}, T(\mathcal{S}))$$

the ∞-category of functors which lift x to $T(\mathcal{S})$ along the projection map $T(\mathcal{S}) \to \mathcal{S}$.

Example 3.11. Taking $\mathsf{C} = *$ in Example 3.10 we obtain an affine ∞-topos $\mathcal{A}^*$ whose ∞-category of points is $\mathcal{S}$. Therefore

$$U(\mathcal{A}^*) \simeq \mathrm{Fun}^{\omega}(T(\mathcal{S}), \mathcal{S})$$

and, for an ∞-groupoid $x \in \mathcal{S}$,

$$p(U_x \mathcal{A}^*) \simeq T_x \mathcal{S} = \mathcal{S}p(\mathcal{S}_{/x})$$

the ∞-category of spectra parameterized over x.

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