

Tangent ∞ -Categories and Goodwillie Calculus

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Abstract

We make precise the analogy between Goodwillie’s calculus of functors in homotopy theory and the differential calculus of smooth manifolds by introducing a higher-categorical framework of which both theories are examples. That framework is an extension to ∞ -categories of the *tangent categories* of Cockett and Cruttwell (introduced originally by Rosický). The basic data of a tangent ∞ -category consist of an endofunctor, that plays the role of the tangent bundle construction, together with various natural transformations that mimic structure possessed by the ordinary tangent bundles of smooth manifolds.

The role of the tangent bundle functor in Goodwillie calculus is played by Lurie’s tangent bundle for ∞ -categories, introduced to generalize the cotangent complexes of André, Quillen and Illusie. We show that Lurie’s construction admits the additional structure maps and satisfies the conditions needed to form a tangent ∞ -category which we refer to as the *Goodwillie tangent structure*.

Cockett and Cruttwell (and others) have started to develop various aspects of differential geometry in the abstract context of tangent categories, and we begin to apply those ideas to Goodwillie calculus. For example, we show that the role of Euclidean spaces in the calculus of manifolds is played in Goodwillie calculus by the stable ∞ -categories. We also show that Goodwillie’s n -excisive functors are the direct analogues of n -jets of smooth maps between manifolds; to state that connection precisely, we develop a notion of tangent $(\infty, 2)$ -category and show that Goodwillie calculus is best understood in that context.

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Introduction

Goodwillie’s calculus of functors, developed in the series of papers [Goo90, Goo91, Goo03], provides a systematic way to apply ideas from ordinary calculus to homotopy theory. For example, central to this functor calculus is the ‘Taylor tower’, an analogue of the Taylor series, which comprises a sequence of ‘polynomial’ approximations to a functor between categories of topological spaces or spectra.

The purpose of this paper is to explore a slightly different analogy, also proposed by Goodwillie, where we view the categories of topological spaces or spectra (or, to be more precise, the corresponding ∞ -categories) as analogues of smooth manifolds, and the functors between those categories as playing the role of smooth maps. Our main goal is to make this analogy precise by introducing a common framework within which both Goodwillie calculus and the calculus of smooth manifolds exist as examples. That framework is the theory of *tangent categories*¹ of Cockett and Cruttwell [CC14], extended to ∞ -categories.

A central object in the calculus of manifolds is the tangent bundle construction: associated to each smooth manifold M is another smooth manifold TM along with a projection $p_M : TM \rightarrow M$ as well as various other structure maps that, among other things, make p_M into a vector bundle for each M . The first step in merging functor calculus with differential geometry is to describe the ‘tangent bundle’ for an ∞ -category, such as that of spaces or spectra.

Fortunately for us, the analogue of the tangent bundle construction in homotopy theory is well-known. In the generality we want in this paper, that construction is given by Lurie in [Lur17, 7.3.1], but the ideas go back at least to work of André [And74] and Quillen [Qui70] on cohomology theories for commutative rings, and Illusie [Ill71] on cotangent complexes in algebraic geometry.

For an object X in an ordinary category \mathcal{C} , we can define the ‘tangent space’ to \mathcal{C} at X to be the category of abelian group objects in the slice category $\mathcal{C}_{/X}$ of objects over X . These abelian group objects are also called *Beck modules* after their introduction by Beck in a 1967 Ph.D. thesis [Bec03], and they were used by Quillen [Qui67] as the coefficients for the definition of cohomology in an arbitrary model category.

Quillen’s approach was refined by Bastera and Mandell [BM05], building on previous work of Bastera [Bas99] on ring spectra, to what is now known as *topological* Quillen homology. In that refinement the abelian group objects are replaced by cohomology theories, or spectra in the sense of stable homotopy theory.

¹The phrase ‘tangent category’ is unfortunately used in the literature to describe two different objects, both of which feature extensively in this paper. We follow Cockett and Cruttwell’s terminology, and use ‘tangent category’ to refer to the broader notion: a category whose objects admit tangent bundles. The notion referred to by, for example, Harpaz-Nuiten-Prasma in the title of [HNP19b] will for us be called the ‘tangent bundle’ on a category (or ∞ -category).

In particular, for an object X in an ∞ -category \mathcal{C} , our *tangent space* to \mathcal{C} at X :

$$(0.1) \quad T_X \mathcal{C} := \mathrm{Sp}(\mathcal{C}_{/X})$$

is the ∞ -category of spectra in the corresponding slice ∞ -category of objects over X . This ∞ -category $T_X \mathcal{C}$ is the ‘stabilization’ of $\mathcal{C}_{/X}$: its best approximation by an ∞ -category that is stable.²

For example, when \mathcal{C} is the ∞ -category of topological spaces, $T_X \mathcal{C}$ can be identified with the ∞ -category of spectra parameterized³ over the topological space X . Basterra and Mandell [BM05] prove that when \mathcal{C} is the ∞ -category of commutative ring spectra, $T_R \mathcal{C}$ is the ∞ -category of R -module spectra. Various other calculations of these tangent spaces have been done recently in a series of papers by Harpaz, Nuiten and Prasma: [HNP19b] (for algebras over operads of spectra), [HNP18] (for ∞ -categories themselves) and [HNP19a] (for $(\infty, 2)$ -categories).

Lurie’s construction [Lur17, 7.3.1.10] collects all of these individual tangent spaces together into a ‘tangent bundle’: for each suitably nice ∞ -category \mathcal{C} there is a functor

$$p_{\mathcal{C}} : T\mathcal{C} \rightarrow \mathcal{C}$$

whose fibre over X is precisely $T_X \mathcal{C}$. One of the aims of this paper is to give substance to the claim that this $p_{\mathcal{C}}$ is analogous to the ordinary tangent bundle $p_M : TM \rightarrow M$ for a smooth manifold M , by presenting both as examples of the notion of ‘tangent category’.

Tangent categories. A categorical framework for the tangent bundle construction on smooth manifolds was first provided by Rosický in [Ros84]. In that work, he describes various structure that can be built on the tangent bundle functor

$$T : \mathrm{Mfld} \rightarrow \mathrm{Mfld}$$

where Mfld denotes the category of smooth manifolds and smooth maps. Of course there is a natural transformation $p : T \rightarrow \mathrm{Id}$ that provides the tangent bundle projection maps. There are also natural transformations $0 : \mathrm{Id} \rightarrow T$, given by the zero section for each tangent bundle, and $+$: $T \times_{\mathrm{Id}} T \rightarrow T$, capturing the additive structure of those vector bundles.

Rosický’s work was largely unused until resurrected in 2014 by Cockett and Cruttwell [CC14] in order to describe connections between calculus on manifolds and structures appearing in logic and computer science, such as the differential λ -calculus of Ehrhard and Regnier [ER03]. Cockett and Cruttwell define a tangent structure on a category \mathbb{X} to consist of an endofunctor $T : \mathbb{X} \rightarrow \mathbb{X}$ together with five natural transformations:

- the *projection* $p : T \rightarrow \mathrm{Id}$
- the *zero section* $0 : \mathrm{Id} \rightarrow T$
- the *addition* $+$: $T \times_{\mathrm{Id}} T \rightarrow T$
- the *flip* $c : T^2 \rightarrow T^2$
- the *vertical lift* $\ell : T \rightarrow T^2$

for which there is a large collection of diagrams that are required to commute. As indicated above, the first three of these natural transformations make TM into a

²An ∞ -category is *stable* when it admits a null object, it has finite limits and colimits, and when pushout and pullback squares coincide; see [Lur17, 1.1] for an extended introduction.

³See [MS06] for parameterized spectra.

bundle of commutative monoids over M , for any object $M \in \mathbb{X}$.⁴ The maps c and ℓ express aspects of the ‘double’ tangent bundle $T^2M = T(TM)$ that are inspired by the case of smooth manifolds: its symmetry in the two tangent directions, and a canonical way of ‘lifting’ tangent vectors to the double tangent bundle.

In addition to the required commutative diagrams, there is one crucial additional condition, referred to by Cockett and Cruttwell [CC18, 2.1] as the ‘universality of the vertical lift’. This axiom states that certain squares of the form

$$(0.2) \quad \begin{array}{ccc} TM \times_M TM & \longrightarrow & T^2M \\ \downarrow & & \downarrow T(p) \\ M & \xrightarrow{0} & TM \end{array}$$

are required to be pullback diagrams in \mathbb{X} . (See 1.12 for a precise statement.) When $\mathbb{X} = \mathbf{Mfld}$ and M is a smooth manifold, this vertical lift axiom can be expressed as a collection of diffeomorphisms

$$T(T_x M) \cong T_x M \times T_x M$$

varying smoothly with $x \in M$.

Thus the vertical lift axiom tells us a familiar fact: that the tangent bundle of a tangent space $T_x M$ (indeed of any Euclidean space) is trivial. Cockett and Cruttwell’s insight was that this axiom is also the key to translating many of the constructions of ordinary differential geometry into the context of abstract tangent categories. Since the publication of [CC14] a small industry has developed around this task, providing, for example:

- a *Lie bracket for vector fields* (described by Rosický in his original work on tangent categories [Ros84] and refined in [CC14]);
- an analogue for *smooth vector bundles*: the ‘differential bundles’ introduced in [CC18];
- notions of *connection*, *torsion* and *curvature*, developed by Cockett and Cruttwell [CC17], also studied by Lucyshyn-Wright [Luc17];
- analogues of *affine spaces*, described by Blute, Cruttwell and Lucyshyn-Wright [BCLW19];
- *differential forms* and *cohomology*, studied by Cruttwell and Lucyshyn-Wright [CLW18]; and
- *Lie algebroids*, studied by the second author and MacAdam [BM19].

In Chapter 11 of this paper we add to this list the notion of ‘ n -jet’ of smooth maps in order to make precise the connection with Goodwillie’s n -excisive functors. We do not address any of these other topics here, though we expect each of them has an analogue for tangent ∞ -categories, which might be worth studying. Note, however, that some of these concepts, such as the Lie bracket, depend on the existence of additive inverses in the tangent bundle, and hence are not applicable in the main example of this paper where those inverses do not exist.

⁴In Rosický’s original work, the addition map was required to have fibrewise negatives so that TM is a bundle of abelian groups. Cockett and Cruttwell relaxed that conditions, and we will take full advantage of this relaxation since Lurie’s tangent bundle does not admit negatives.

Tangent ∞ -categories. Our paper is split into two parts, and the first part focuses on extending the theory of tangent categories to the ∞ -categorical context. The principal goal of this part is Definition 3.2 which introduces the notion of tangent structure on an ∞ -category \mathbb{X} , and which recovers Cockett and Cruttwell’s definition when \mathbb{X} is an ordinary category.

A challenge in making this definition is that the large array of commutative diagrams listed by Cockett and Cruttwell in their definition of tangent category would require an even larger array of higher cohering homotopies if translated directly to the ∞ -categorical framework. Fortunately, work of Leung [Leu17] provides a more conceptual definition of tangent category based on a category of ‘Weil-algebras’ already used in algebraic geometry and synthetic differential geometry.

In Definition 1.4 we describe a symmetric monoidal category $\mathbb{W}eil_1$ whose objects are certain augmented commutative semi-rings of the form

$$\mathbb{N}[x_1, \dots, x_n]/(x_i x_j)$$

where the relations are given by some set of quadratic monomials that includes the squares x_i^2 . The monoidal structure on $\mathbb{W}eil_1$ is given by tensor product. These objects form only a subset of the more general Weil-algebras introduced by Weil in [Wei53] to study tangent vectors on manifolds in terms of infinitesimals. Note that the appearance of semi-rings in this definition, rather than algebras over \mathbb{Z} or a field, corresponds to the fact that the additive structure in our tangent bundles is not required to admit fibrewise negation.

Leung’s main result [Leu17, 14.1] is that the structure of a tangent category \mathbb{X} is precisely captured by a monoidal functor

$$(0.3) \quad T^\otimes : \mathbb{W}eil_1 \rightarrow \text{End}(\mathbb{X})$$

from $\mathbb{W}eil_1$ to the category of endofunctors on \mathbb{X} under composition, or equivalently, to an *action* of $\mathbb{W}eil_1$ on \mathbb{X} . This action is subject to the additional condition that certain pullbacks be preserved by T^\otimes ; in particular, this condition provides the ‘universality of vertical lift’ axiom referred to in (0.2).

The simplest non-trivial example of a Weil-algebra is the ‘dual numbers’ object $W = \mathbb{N}[x]/(x^2)$ which, under Leung’s formulation, corresponds to the ordinary tangent bundle functor $T : \mathbb{X} \rightarrow \mathbb{X}$. Evaluating the functor T^\otimes on morphisms in $\mathbb{W}eil_1$, i.e. the homomorphisms between Weil-algebras, provides the various natural transformations that make up the tangent structure. For example $\mathbb{W}eil_1$ encodes the structure that makes each projection map $TM \rightarrow M$ into a bundle of commutative monoids over M .

In order to generalize tangent structures to ∞ -categories, we construct in Chapter 2 an ∞ -categorical version of $\mathbb{W}eil_1$, which we denote simply $\mathbb{W}eil$. The monoidal ∞ -category $\mathbb{W}eil$, which has $\mathbb{W}eil_1$ as its homotopy category, includes certain non-trivial 2-isomorphisms that encode bundles of E_∞ -monoids in place of the strictly commutative and associative monoids appearing in ordinary tangent categories.⁵

⁵In the original version of this paper we used the 1-category $\mathbb{W}eil_1$ in our definition of tangent ∞ -structure. We are grateful to Thomas Nikolaus and Maxime Ramzi for pointing out that, with this definition, our construction of the Goodwillie tangent structure entails strictly commutative structures on the ∞ -category of spectra which cannot exist. There was a mistake in our original construction; specifically the natural transformation α in the previous version of Definition 8.18 was not sufficiently well-defined. That comment led us to the introduction of the monoidal ∞ -category $\mathbb{W}eil$ defined in Chapter 2.

Our Definition 3.2 of tangent structure on an ∞ -category \mathbb{X} then follows the same format as Leung’s characterization of tangent categories; it is a functor of monoidal ∞ -categories of the form (0.3), but with Weil_1 replaced by the Weil , and that preserves those same pullbacks (though now with pullback understood in the ∞ -categorical sense). Under this definition any ordinary tangent category, such as Mfld with its usual tangent bundle construction, is also a tangent ∞ -category with structure map given by composing the relevant map (0.3) with the projection $\text{Weil} \rightarrow \text{Weil}_1$ from Weil to (the nerve of) its homotopy category.

There are, however, tangent ∞ -categories that do not arise from an ordinary tangent category. One example is given by the ‘derived smooth manifolds’ of [Spi10]. There, Spivak defines an ∞ -category $\mathbb{D}\text{Mfld}$ that contains Mfld as a full subcategory, but which admits all pullbacks, not only those along transverse pairs of smooth maps. We show in Proposition 3.15 that the tangent structure on Mfld extends naturally to $\mathbb{D}\text{Mfld}$ using a universal property described by Carchedi and Steffens [CS19].

Tangent $(\infty, 2)$ -categories and other objects. The definition of tangent structure on an ∞ -category can easily be extended to a wide range of other types of objects, and we examine this generalization in Chapter 6. Let \mathbb{X} be an object in an $(\infty, 2)$ -category \mathbf{C} which, for the purposes of this introduction, one may view simply as a category enriched in ∞ -categories. Then \mathbb{X} admits a monoidal ∞ -category $\text{End}_{\mathbf{C}}(\mathbb{X})$ of endomorphisms. We thus define (in 6.11) a tangent structure on the object \mathbb{X} to be a monoidal functor

$$T : \text{Weil} \rightarrow \text{End}_{\mathbf{C}}(\mathbb{X})$$

that preserves the appropriate pullbacks. This definition recovers our notion of tangent ∞ -category (and hence of tangent category) when \mathbf{C} is the $(\infty, 2)$ -category of ∞ -categories. For example, taking \mathbf{C} to be a suitable $(\infty, 2)$ -category of $(\infty, 2)$ -categories, we also obtain a notion of tangent $(\infty, 2)$ -category.

The Goodwillie tangent structure on differentiable ∞ -categories. In the second part of this paper, we construct a specific tangent ∞ -category for which the tangent bundle functor is equivalent to that defined by Lurie, and which encodes the theory of Goodwillie calculus. The existence of this tangent structure (which we refer to as the *Goodwillie tangent structure*) justifies the analogy between functor calculus and the calculus of smooth manifolds.

The underlying ∞ -category for this tangent structure is $\text{Cat}_{\infty}^{\text{diff}}$: a subcategory of Lurie’s ∞ -category of ∞ -categories [Lur09a, 3.0.0.1]. The objects in $\text{Cat}_{\infty}^{\text{diff}}$ are those ∞ -categories \mathcal{C} that are *differentiable* in the sense introduced in [Lur17, 6.1.1.6]: those \mathcal{C} that admit finite limits and sequential colimits, which commute. This condition is satisfied by many ∞ -categories of interest including any compactly generated ∞ -category, such as that of topological spaces, and any ∞ -topos. The morphisms in the ∞ -category $\text{Cat}_{\infty}^{\text{diff}}$ are those functors between differentiable ∞ -categories that preserve sequential colimits.

The tangent bundle construction from [Lur17, 7.3.1.10] can be described explicitly as the functor $T : \text{Cat}_{\infty}^{\text{diff}} \rightarrow \text{Cat}_{\infty}^{\text{diff}}$ given by

$$T\mathcal{C} := \text{Exc}(\mathcal{S}_{\text{fin},*}, \mathcal{C})$$

where $\mathcal{S}_{\text{fin},*}$ denotes the ∞ -category of finite pointed spaces, and $\text{Exc}(-, -)$ denotes the ∞ -category of functors that are *excisive* in the sense of Goodwillie (i.e. map

pushouts in $\mathcal{S}_{\text{fin},*}$ to pullbacks in \mathcal{C}). With this definition in mind, the Goodwillie tangent structure on $\text{Cat}_{\infty}^{\text{diff}}$ consists of the following natural transformations:

- the *projection* map $p : T\mathcal{C} \rightarrow \mathcal{C}$ is evaluation at the null object:

$$L \mapsto L(*);$$

- the *zero section* $0 : \mathcal{C} \rightarrow T\mathcal{C}$ maps an object X of \mathcal{C} to the constant functor with value X ;
- *addition* $+ : T\mathcal{C} \times_{\mathcal{C}} T\mathcal{C} \rightarrow T\mathcal{C}$ is the fibrewise product:

$$(L_1, L_2) \mapsto L_1(-) \times_{L_1(*)=L_2(*)} L_2(-);$$

- identifying $T^2\mathcal{C}$ with the ∞ -category of functors $\mathcal{S}_{\text{fin},*} \times \mathcal{S}_{\text{fin},*} \rightarrow \mathcal{C}$ that are excisive in each variable individually, the *flip* $c : T^2\mathcal{C} \rightarrow T^2\mathcal{C}$ is the symmetry in those two variables:

$$L \mapsto [(X, Y) \mapsto L(Y, X)];$$

- the *vertical lift* map $\ell : T\mathcal{C} \rightarrow T^2\mathcal{C}$ is precomposition with the smash product:

$$L \mapsto [(X, Y) \mapsto L(X \wedge Y)].$$

A complete definition of the Goodwillie tangent structure requires the construction of a monoidal functor

$$T : \mathbb{W}\text{eil} \rightarrow \text{End}(\text{Cat}_{\infty}^{\text{diff}}).$$

For a Weil-algebra of the form

$$A = \mathbb{N}[x_1, \dots, x_n]/(x_i x_j)$$

we define the corresponding endofunctor $T^A : \text{Cat}_{\infty}^{\text{diff}} \rightarrow \text{Cat}_{\infty}^{\text{diff}}$ by

$$T^A(\mathcal{C}) := \text{Exc}^A(\mathcal{S}_{\text{fin},*}^n, \mathcal{C}) \subseteq \text{Fun}(\mathcal{S}_{\text{fin},*}^n, \mathcal{C}),$$

the full subcategory consisting of those functors $\mathcal{S}_{\text{fin},*}^n \rightarrow \mathcal{C}$ that satisfy the property of being ‘ A -excisive’ (see Definition 8.1). When $A = \mathbb{N}[x]/(x^2)$, A -excisive is excisive, and we recover Lurie’s definition of the tangent bundle $T\mathcal{C}$. We define the functor T on a morphism ϕ in $\mathbb{W}\text{eil}$ via precomposition with a certain functor

$$\tilde{\phi} : \mathcal{S}_{\text{fin},*}^{n'} \rightarrow \mathcal{S}_{\text{fin},*}^n$$

whose construction mimics the algebra homomorphism underlying ϕ ; see Definition 8.12 for details.

In Chapter 8 we prove the key homotopy-theoretic results needed to check that the definitions above do indeed form a tangent structure. Principal among those is the vertical lift axiom analogous to that in (0.2) above; this axiom is verified in Proposition 8.36, and involves in detail the classification of multilinear functors in Goodwillie calculus, and splitting results for functors with values in a stable ∞ -category. It is there that the technical heart of the construction of the Goodwillie tangent structure lies.

The full definition of T as a monoidal functor between monoidal ∞ -categories is rather involved and relies on a model for $\text{Cat}_{\infty}^{\text{diff}}$ based on ‘relative’ ∞ -categories; see [MG19]. The specifics of this construction are in Chapter 9. The reader interested in understanding the basic idea of our construction, rather than the intricate details, should focus on Chapter 8 where the most important of the underlying definitions are provided.

Tangent functors, differential objects and jets. Much of this paper is concerned with the definition of tangent ∞ -category and the construction of the Goodwillie tangent structure on the ∞ -category $\mathcal{C}at_{\infty}^{\text{diff}}$. However, we also begin the task of extending the current tangent category literature to the ∞ -categorical context. In this paper, we focus on three specific aspects of that theory: functors between tangent categories, differential objects, and jets.

As with any categorical structure, it is crucial to identify the appropriate morphisms. Cockett and Cruttwell introduced in [CC14, 2.7] two notions of ‘morphism of tangent structure’ (‘lax’ and ‘strong’), and in Chapter 4 we extend those notions to tangent ∞ -categories. Briefly, a tangent functor between tangent ∞ -categories is a functor that commutes (up to higher coherent equivalences, in the strong case, or up to coherent natural transformations, in the lax case) with the corresponding actions of Weil. We give an explicit model for these higher coherences in Definition 4.6 which, using work of Garner [Gar18], reduces to Cockett and Cruttwell’s definition in the case of ordinary categories.

Differential objects were introduced by Cockett and Cruttwell [CC14, 4.8] in order to axiomatize the role of Euclidean spaces in the theory of smooth manifolds: these are the objects that play the role of tangent *spaces*. In Chapter 5 we describe our extension of the theory of differential objects to tangent ∞ -categories. Our description provides a new perspective on differential objects even in the setting of ordinary tangent categories; see Proposition 5.8.

Another role for differential objects is in making the connection between tangent categories and the ‘cartesian differential categories’ of Blute, Cockett and Seely [BCS09]. Roughly speaking, a cartesian differential category is a tangent category in which every object has a canonical differential structure; for example, the subcategory of $\mathbb{M}fd$ whose objects are the Euclidean spaces \mathbb{R}^n . We show that a similar relationship (Theorem 5.29) also holds in the tangent ∞ -category setting after passage to homotopy categories.

In Chapter 10 we analyze the notion of differential object in the specific context of the Goodwillie tangent structure on $\mathcal{C}at_{\infty}^{\text{diff}}$. It is not hard to see that the differential objects in $\mathcal{C}at_{\infty}^{\text{diff}}$ are precisely the *stable* ∞ -categories. This result is not at all surprising; the role of stabilization is built into our tangent structure via the tangent spaces described in (0.1). It does, however, confirm Goodwillie’s intuition that categories of spectra should be viewed, from the point of view of functor calculus, as analogues of Euclidean spaces.

We also deduce the existence of a cartesian differential category whose objects are the stable ∞ -categories and whose morphisms are natural equivalence classes of functors. This result extends work of the first author and Johnson, Osborne, Riehl, and Tebbe [BJO⁺18] which describes a similar construction for (chain complexes of) abelian categories in the context of Johnson and McCarthy’s ‘abelian functor calculus’ variant of Goodwillie’s theory [JM04]. In fact, the paper [BJO⁺18] provided much of the inspiration for our development of tangent ∞ -categories and for the construction of the Goodwillie tangent structure.

In Chapter 11 we turn to the notion of ‘*n*-jet’ of a morphism which does not appear explicitly in the tangent category literature, though it has been studied in the context of synthetic differential geometry, e.g. see [Koc10, 2.7]. In our case, the importance of *n*-jets is that they correspond in the Goodwillie tangent

structure on $\mathcal{C}\text{at}_\infty^{\text{diff}}$ to the n -excisive functors, i.e. Goodwillie’s analogues of degree n polynomials.

In an arbitrary tangent ∞ -category \mathbb{X} , we say that two morphisms $F, G : \mathcal{C} \rightarrow \mathcal{D}$ determine the same n -jet at a (generalized) point $x \in \mathcal{C}$ if they induce equivalent maps on the n -fold tangent spaces $T_x^n \mathcal{C}$ at x . For smooth manifolds, this definition recovers the ordinary notion of n -jet; the equivalence class of smooth maps that agree to order n in a neighbourhood of the point x . For ∞ -categories, we then prove an analogous result (Theorem 11.3) which says that a natural transformation $\alpha : F \rightarrow G$ between two functors $\mathcal{C} \rightarrow \mathcal{D}$ induces an equivalence $P_n^x F \xrightarrow{\sim} P_n^x G$ between Goodwillie’s n -excisive approximations at x if and only if α induces an equivalence on $T_x^n \mathcal{C}$.

The significance of the previous result is that it shows that the notion of n -excisive equivalence, and hence n -excisive functor, can be recovered directly from the Goodwillie tangent structure. To make proper sense of this claim though, we need to be able to talk about non-invertible natural transformations between functors of ∞ -categories. This observation reveals that Goodwillie calculus is better understood in the context of tangent structures on an $(\infty, 2)$ -category.

In Theorem 12.15 we show that there is an $(\infty, 2)$ -category $\mathcal{C}\text{AT}_\infty^{\text{diff}}$ of differentiable ∞ -categories which admits a Goodwillie tangent structure extending that on $\mathcal{C}\text{at}_\infty^{\text{diff}}$. This tangent $(\infty, 2)$ -category completely encodes Goodwillie calculus and the notion of Taylor tower.

Conjectures and connections. Lurie introduced the tangent bundle $T\mathcal{C}$ not directly in relation to Goodwillie calculus but as part of the development of deformation theory in an ∞ -category \mathcal{C} ; see [Lur17, 7.4]. That theory is controlled by the cotangent complex functor, a certain section

$$L : \mathcal{C} \rightarrow T\mathcal{C}$$

of the tangent bundle projection map; see [Lur17, 7.3.2.14]. The functor L does not appear to have an analogue in the general theory of tangent categories, and it has not played any role in this paper. Nonetheless it would be interesting to explore what aspects of the Goodwillie tangent structure allow for the cotangent complex and corresponding deformation theory to be developed.

There are other topics and questions regarding the Goodwillie tangent structure which we would like to have addressed in this paper. One such question is the extent to which the Goodwillie tangent structure on $\mathcal{C}\text{at}_\infty^{\text{diff}}$ is unique. We believe that indeed it is the unique (up to contractible choice) tangent ∞ -category which extends Lurie’s tangent bundle construction, but we do not try to give a proof of that conjecture here.

Another topic concerns what are known as ‘representable’ tangent categories. It was observed by Rosický that a model for synthetic differential geometry gives rise to a tangent category in which the tangent bundle functor is represented by an object with so-called ‘infinitesimal’ structure; see [CC14, 5.6]. It appears relatively easy to extend the definition of representable tangent structure to ∞ -categories, but also to prove that the Goodwillie tangent structure is *not* representable in this sense. In fact, Lurie’s tangent bundle functor itself is not representable.

However, if we restrict the Goodwillie tangent structure to the subcategory of $\mathcal{C}\text{at}_\infty^{\text{diff}}$ consisting of ∞ -toposes, e.g. see [Lur09a, 6.1.0.4], and the left exact colimit-preserving functors, then this restricted tangent structure is *dual*, in the

sense of [CC14, 5.17], to a representable tangent structure on Topos_∞ (the ∞ -category of ∞ -toposes and geometric morphisms). The representing object for that tangent structure is the ∞ -topos $T(\mathcal{S}) = \text{Exc}(\mathcal{S}_{\text{fin},*}, \mathcal{S})$, whose objects are parameterized spectra over arbitrary topological spaces, with infinitesimal structure arising directly from the Goodwillie tangent structure. Details of these claims are worked out by the third author in [Chi21]. We wonder if such structure is related to work of Anel, Biedermann, Finster and Joyal [ABFJ18] on Goodwillie calculus for ∞ -toposes.

Earlier in this introduction we listed various topics from ordinary differential geometry that have been developed in the abstract setting of tangent categories, including vector bundles, connections, and curvature. For each of these topics, or any others that can be formulated in an abstract tangent category, we can speculate on what form they take in the Goodwillie tangent structure. Some ideas along those lines appear at the end of this paper in a section on [Proposals for Future Work](#).

There are two other directions for generalization that seem particularly worthy of exploration. One is to replace $\text{Cat}_\infty^{\text{diff}}$ in the Goodwillie tangent structure with a different ∞ -cosmos in the sense of Riehl and Verity [RV22, 1.2]. The other is to look for tangent ∞ -categories that encode other versions of functor calculus, such as the ‘manifold calculus’ of Goodwillie and Weiss [Wei99, GW99], or the ‘orthogonal calculus’ of Weiss [Wei95]. Other lines of inquiry arise from finding counterparts in tangent categories of concepts that are already established in Goodwillie calculus, such as Heuts’s work [Heu21] on Goodwillie towers for (pointed compactly-generated) ∞ -categories, instead of functors, and work of Greg Arone and the third author [AC11] on chain rules and the role of operads in functor calculus.

Finally, we might speculate on how the Goodwillie tangent structure of this paper fits into the much bigger programme of ‘higher differential geometry’ developed by Schreiber [Sch13, 4.1], or into the framework of homotopy type theory [Pro13], though we don’t have anything concrete to say about these possible connections.

Background on ∞ -categories; notation and conventions. This paper is written largely in the language of ∞ -categories, as developed by Lurie in the books [Lur09a] and [Lur17], based on the quasi-categories of Boardman and Vogt [BV73]. However, for much of the paper, details of that theory are not particularly important, and other models for $(\infty, 1)$ -categories could easily be used instead, especially if the reader is interested only in the main ideas of this work rather than the technical details.

All the basic concepts needed to define tangent ∞ -categories, and to describe the underlying data of the Goodwillie tangent structure, will be familiar to readers versed in ordinary category theory. Those ideas include limits and colimits (in particular, pullbacks and pushouts), adjunctions, monoidal structures, and functor categories. To follow the details of our constructions, however, the reader will need a close acquaintance with simplicial sets and, especially, their relationship to categories via the nerve construction.

Throughout this paper, we take the perspective that categories, ∞ -categories, and indeed $(\infty, 2)$ -categories, are all really the same sort of thing, namely simplicial sets (sometimes with additional data). We typically do not distinguish notationally between any of these types of object. In particular, we identify a category \mathbb{X} with its nerve, an ∞ -category. In a few places we will consider simplicially-enriched

categories, which we also usually identify with simplicial sets via the simplicial (or homotopy coherent) nerve [Lur09a, 1.1.5.5].

One exception to this convention is the category of simplicial sets itself which we denote as \mathbf{Set}_Δ , and some related categories (such as ‘marked’ or ‘scaled’ simplicial sets) which we introduce in the course of this paper. We will make some use of the various model structures on these categories, including the Quillen and Joyal model structures on \mathbf{Set}_Δ . There are ∞ -categories associated to these model categories, but we will use separate notation to denote those ∞ -categories when we need to invoke them.

A functor $F : \mathbb{X} \rightarrow \mathbb{Y}$ between two ∞ -categories (or $(\infty, 2)$ -categories) is simply a map of simplicial sets (possibly required to respect additional data). When \mathbb{X} is an ordinary category, such F can be viewed as a ‘homotopy coherent’ diagram in the ∞ -category \mathbb{Y} . If \mathbb{Y} is also an ordinary category, then F is an ordinary functor from \mathbb{X} to \mathbb{Y} .

For any two simplicial sets A, B , we write $\mathbf{Fun}(A, B)$ for the simplicial set whose n -simplexes are the simplicial maps $\Delta^n \times A \rightarrow B$, i.e. the ordinary simplicial mapping object. When B is an ∞ -category, so is $\mathbf{Fun}(A, B)$, and in that case we refer to $\mathbf{Fun}(A, B)$ as the ‘ ∞ -category of functors from A to B ’.

One of the most confusing aspects of our theory is that ∞ -categories (and $(\infty, 2)$ -categories) play several different roles in this paper at different ‘levels’:

- (1) we can define **tangent objects** in any (potentially very large) $(\infty, 2)$ -category \mathbf{C} (Definition 6.11);
- (2) taking \mathbf{C} to be the $(\infty, 2)$ -category of (large) ∞ -categories \mathbf{Cat}_∞ , we get a notion of **tangent structure on** a specific ∞ -category $\mathbb{C} \in \mathbf{C}$ (Definition 3.2);
- (3) taking \mathbb{C} to be an ∞ -category of (smaller) ∞ -categories, such as $\mathbf{Cat}_\infty^{\text{diff}}$, we get the **tangent bundle of** a specific ∞ -category $\mathbb{C} \in \mathbb{C}$ (Definition 7.12).

We can also take \mathbb{C} in (3) to be an ∞ -category \mathbf{Cat}_∞ of (even smaller) ∞ -categories, in which case we would also have the **tangent space at** an ∞ -category Y , i.e. $T_Y \mathbb{C}$. These tangent spaces are studied in [HNP18] but do not play any particular role for us here.

We distinguish between these three uses for ∞ - and $(\infty, 2)$ -categories by applying the fonts $\mathbf{C}, \mathbb{C}, \mathcal{C}$ as indicated in the list above. In particular, we use these different fonts to signify the size restrictions that are implicit in our hierarchy. To be precise we assume, where necessary, the existence of inaccessible cardinals that determine the different ‘sizes’ of the ∞ -categories described in (1), (2) and (3) above; see also [Lur09a, 1.2.15] for a discussion of this foundational issue. Beyond the requirement to keep these three levels separate, size issues do not play any significant role in this paper.

One final (and important) comment on notation, especially for readers familiar with the papers of Cockett and Cruttwell: in this paper we use ‘algebraic’ order for writing composition, as opposed to the diagrammatic order employed in many papers in the tangent category literature. So, for morphisms $f : A \rightarrow B$ and $g : B \rightarrow C$, we write gf for the composite morphism $A \rightarrow C$. Because of this choice, some of the expressions we use look different to those appearing in a corresponding place in [CC14].

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Part 1

**Tangent Structures on
 ∞ -Categories**

Weil-Algebras and Tangent Categories

The goal of Part 1 is to extend the notion of tangent category of Cockett and Cruttwell to an ∞ -categorical context, and to begin the study of these ‘tangent ∞ -categories’.

We refer the reader to the paper of Cockett and Cruttwell [CC14] for a detailed introduction to the theory of tangent categories. The definition that we use in this paper, however, is based on an alternative characterization of tangent structure, due to Leung [Leu17, 14.1]. That characterization is in terms of a monoidal category of ‘Weil-algebras’, and in Chapter 1 we recall that category and the corresponding notion of tangent structure on a category.

In Chapter 2 we turn to ∞ -categories. Our first goal is to define an ∞ -categorical version of the monoidal category of Weil-algebras, which we use to give a definition of tangent ∞ -category in Chapter 3. We start to develop the theory of tangent ∞ -categories in Chapter 4 with a definition of tangent functor, the appropriate notion of morphism between tangent ∞ -categories. In Chapter 5 we consider the notion of ‘differential object’ in a tangent structure, also due to Cockett and Cruttwell, and extend that notion to tangent ∞ -categories.

In the last topic in this part, Chapter 6, we extend the definition of tangent structure to $(\infty, 2)$ -categories and other kinds of objects.

We start by recalling the Weil-algebras used in Leung’s description of tangent structure.

DEFINITION 1.1. Let $\mathbb{C}\text{Rig}$ denote the category of commutative semi-rings (or rigs) as follows:

- an object of $\mathbb{C}\text{Rig}$ is a set A equipped with two binary operations, which we refer to as addition and multiplication, each of which makes A into a commutative monoid (with corresponding identity elements 0 and 1), which satisfy the distributive law, and such that $0 \cdot a = 0$ for all $a \in A$;
- a morphism of $\mathbb{C}\text{Rig}$ is a function between underlying sets which commutes with addition and multiplication, and preserves 0 and 1;
- the identity morphism 1_A is the identity function on A ;
- composition in $\mathbb{C}\text{Rig}$ is composition of functions.

EXAMPLE 1.2. The set \mathbb{N} of natural numbers, including 0, is an object in $\mathbb{C}\text{Rig}$. For each finite set J , let W^J denote the object whose elements are formal sums

$$n + \sum_{j \in J} (n_j)j$$

with $n, n_j \in \mathbb{N}$, with multiplication determined by the relation $j \cdot j' = 0$ for all $j, j' \in J$. When $J = \{x_1, \dots, x_n\}$ (or some other ordered set of n generators), we

write

$$W^n := \mathbb{N}[x_1, \dots, x_n]/(x_i x_j)_{i,j=1}^n$$

for the corresponding object. We also denote $W^1 = \mathbb{N}[x]/(x^2)$ just as W . When J is the empty set, $W^J = \mathbb{N}$.

DEFINITION 1.3. There is a monoidal structure on $\mathbb{C}\mathbb{R}\text{ig}$ given by the tensor product \otimes with unit object \mathbb{N} . By [Sch01, 4.3] we can choose a model for the tensor product which makes $\mathbb{C}\mathbb{R}\text{ig}$ into a strict monoidal category (i.e. strictly associative, with \mathbb{N} as a strict unit). We assume such a choice has been fixed, the details of which are unimportant.

DEFINITION 1.4. Let Weil_1 be the full subcategory of $\mathbb{C}\mathbb{R}\text{ig}$ whose objects are those of the form

$$W^{J_1} \otimes \dots \otimes W^{J_r}$$

for some finite sequence of finite sets J_1, \dots, J_r . We include the case of the empty sequence in which case this tensor product is equal to \mathbb{N} , the unit for the monoidal structure. We refer to these objects as *Weil-algebras*, and to the morphisms in Weil_1 as *morphisms of Weil-algebras*.

Clearly Weil_1 is a strict monoidal category under the tensor product. This category is (monoidally) equivalent to that denoted $\mathbb{N}\text{-Weil}_1$ in [Leu17].

REMARK 1.5. Let $A = W^{J_1} \otimes \dots \otimes W^{J_r}$ be an object of Weil_1 . Then A is generated (as a semi-ring) by tensors of the form $a_1 \otimes \dots \otimes a_r$ in which one factor is an element of the appropriate J_i and all other factors are equal to 1. It will be convenient to have a simpler notation for referring to these generators, and we will usually denote them x_1, \dots, x_n , where $n = n_1 + \dots + n_r$ and $n_i = |J_i|$ is the number of generators of W^{J_i} . For small numbers of generators we might use x, y, z , or, if we are discussing multiple Weil-algebras, we might use y_1, \dots, y_m to make it clear which object the generators refer to.

The Weil-algebra A can then be written

$$\mathbb{N}[x_1, \dots, x_n]/(x_i x_j \mid i \sim j)$$

where \sim is an equivalence relation on the set $\{1, \dots, n\}$ whose equivalence classes are in bijection with the sets J_1, \dots, J_r . We say that the Weil-algebra A has n *generators*, and we will often identify A by the ordered partition $n = n_1 + \dots + n_r$. Strictly speaking though, an object of Weil_1 requires the specification of the sets J_1, \dots, J_r , not merely their sizes. However, we will often ignore this extra information and write simply $A = W^{n_1} \otimes \dots \otimes W^{n_r}$.

The additive structure on the Weil-algebra A has a basis consisting of the nonzero *monomials*, i.e. those elements which are products of the generators x_1, \dots, x_n . In terms of the underlying tensors, the monomials in $A = W^{J_1} \otimes \dots \otimes W^{J_r}$ are precisely the tensors of the form $j_1 \otimes \dots \otimes j_r$ where for each i , either $j_i \in J_i$ or $j_i = 1$. Let M_A be the finite set whose elements are these nonzero monomials.

REMARK 1.6. A morphism $A \rightarrow A'$ in Weil_1 is determined by the images $\phi(x_i)$ of each of the generators x_i of A . Each $\phi(x_i)$ can be written uniquely as a sum of nonzero monomials in A' . For example, a morphism

$$\mathbb{N}[x, y]/(x^2, xy, y^2) \rightarrow \mathbb{N}[x, y]/(x^2, y^2)$$

might be determined by setting

$$\phi(x) = x + x + xy + y, \quad \phi(y) = xy + xy + xy.$$

Not all choices of sums of monomials determine a morphism: in the above example we must check also that $\phi(x)\phi(y) = \phi(xy) = \phi(0) = 0$, which is the case.

There are certain pullback squares in \mathbb{Weil}_1 that play a crucial role in the definition of tangent structure.

LEMMA 1.7 ([Leu17, 3.14]). *For finite sets J, J' , there is a pullback square in \mathbb{Weil}_1 of the form*

$$\begin{array}{ccc} W^{J \sqcup J'} & \longrightarrow & W^J \\ \downarrow & & \downarrow \\ W^{J'} & \longrightarrow & \mathbb{N} \end{array}$$

in which the horizontal and vertical maps are given by $j' \mapsto 0$ (for $j' \in J'$) and $j \mapsto 0$ (for $j \in J$) respectively. We refer to these diagrams as the foundational pullbacks in \mathbb{Weil}_1 .

LEMMA 1.8. *There is a pullback square in \mathbb{Weil}_1 of the form*

$$\begin{array}{ccc} W^2 & \xrightarrow{\mu} & W \otimes W \\ \downarrow \epsilon & & \downarrow 1_W \otimes \epsilon \\ \mathbb{N} & \xrightarrow{\eta} & W \end{array}$$

where $\mu : \mathbb{N}[x, y]/(x^2, xy, y^2) \rightarrow \mathbb{N}[a, b]/(a^2, b^2)$ is given by

$$\mu(x) = ab, \quad \mu(y) = b.$$

We refer to this square as the vertical lift pullback in \mathbb{Weil}_1 . Collectively, the pullback diagrams in Lemmas 1.7 and 1.8 are the tangent pullbacks.

PROOF. A cone over the maps $1_W \otimes \epsilon$ and η consists of a morphism

$$\phi : \mathbb{N}[z_1, \dots, z_k]/(z_i z_j \mid i \sim j) \rightarrow \mathbb{N}[a, b]/(a^2, b^2)$$

such that each $\phi(z_i)$ is a sum of monomials ab and b (but not a). The corresponding lift

$$\tilde{\phi} : \mathbb{N}[z_1, \dots, z_k]/(z_i z_j \mid i \sim j) \rightarrow \mathbb{N}[x, y]/(x^2, xy, y^2)$$

is given by replacing ab with x and b with y in the formula for each $\phi(z_i)$. \square

We can now give Leung's characterization of a tangent structure on a category \mathbb{X} , which we will use as our definition.

DEFINITION 1.9 ([Leu17, 14.1]). Let \mathbb{X} be a category, and let $\text{End}(\mathbb{X})$ be the strict monoidal category of endofunctors $\mathbb{X} \rightarrow \mathbb{X}$, and their natural transformations, under composition. A *tangent structure* on \mathbb{X} is a strict monoidal functor

$$T : (\mathbb{Weil}_1, \otimes, \mathbb{N}) \rightarrow (\text{End}(\mathbb{X}), \circ, \text{Id})$$

for which the underlying functor $T : \mathbb{Weil}_1 \rightarrow \text{End}(\mathbb{X})$ preserves the tangent pullbacks of Lemmas 1.7 and 1.8.

A *tangent category* (\mathbb{X}, T) consists of a category \mathbb{X} and a tangent structure T on \mathbb{X} .

REMARKS 1.10. There are several minor, and unimportant, differences between our definition of tangent structure and Leung’s formulation:

- (1) The statement of Leung’s theorem [Leu17, 14.1] is that tangent structures correspond to *strong* monoidal functors, yet the proof therein actually constructs a strict monoidal functor T for each tangent structure. It will follow from Lemma 3.19 that any strong monoidal functor $\text{Weil}_1 \rightarrow \text{End}(\mathbb{X})$ is equivalent to a strict monoidal functor (an observation also made by Garner [Gar18, Thm. 7]).
- (2) Leung requires that the tangent structure functor T also preserve pullback squares in Weil_1 of the form

$$\begin{array}{ccc} A \otimes W^{J \sqcup J'} & \longrightarrow & A \otimes W^J \\ \downarrow & & \downarrow \\ A \otimes W^{J'} & \longrightarrow & A \end{array}$$

which he includes in the definition of ‘foundational’ pullback. However, it is sufficient to assume that T preserves those pullbacks in the case $A = \mathbb{N}$, since the more general case follows from that assumption.

- (3) Leung uses a certain equalizer in place of the vertical lift pullback of Lemma 1.8. Cockett and Cruttwell demonstrated the equivalence of these two approaches to the definition of tangent structure in [CC14, 2.12], and focus on the pullback condition in their later work, see [CC18, 2.1].

REMARK 1.11. The strict monoidal functor $T : \text{Weil}_1 \rightarrow \text{End}(\mathbb{X})$ can be described equivalently via an action map

$$\text{Weil}_1 \times \mathbb{X} \rightarrow \mathbb{X}$$

which makes \mathbb{X} into a ‘module’ over the monoidal category Weil_1 . We typically denote this action map also by T and move freely between the two descriptions of a tangent structure.

For each Weil-algebra A , a tangent structure on \mathbb{X} provides for an endofunctor which we denote $T^A : \mathbb{X} \rightarrow \mathbb{X}$. In particular, when $A = W = \mathbb{N}[x]/(x^2)$, we have an endofunctor $T^W : \mathbb{X} \rightarrow \mathbb{X}$ which we refer to as the *tangent bundle functor* of the tangent structure. We often overload the notation still further and denote T^W also by T since in the standard example of smooth manifolds this functor is the ordinary smooth tangent bundle.

We now recall how Definition 1.9 reduces to Cockett and Cruttwell’s original definition of tangent category [CC18, 2.1].

REMARK 1.12. Let $T : \text{Weil}_1 \rightarrow \text{End}(\mathbb{X})$ be a tangent structure on a category \mathbb{X} . The functors $T^A : \mathbb{X} \rightarrow \mathbb{X}$ for Weil-algebras A are determined by the single functor $T = T^W : \mathbb{X} \rightarrow \mathbb{X}$ in the following way:

- since T is monoidal, we have, for the unit object \mathbb{N} of the monoidal structure on Weil_1 :

$$T^{\mathbb{N}} = I$$

the identity functor on \mathbb{X} ;

- since the tangent structure is required to preserve the foundational pullbacks, we have, for any positive integers n :

$$T_n(X) := T^{W^n}(X) \cong T(X) \times_X \cdots \times_X T(X);$$

the wide pullback of n copies of the ‘projection’ map $p_X : T(X) \rightarrow X$ corresponding to the unique map $\epsilon : W \rightarrow \mathbb{N}$; for an arbitrary finite set J , we also write $T_J := T^{W^J}$;

- since any Weil-algebra $A \in \text{Weil}_1$ is of the form $W^{J_1} \otimes \cdots \otimes W^{J_r}$, the monoidal condition then implies

$$T^A = T_{J_1} \cdots T_{J_r}.$$

We also write $T^n := T \cdots T$ for the n -fold composite of T , which corresponds to the Weil-algebra $W \otimes \cdots \otimes W$.

The main content of Leung’s result is that the values of a tangent structure T on morphisms in Weil_1 are determined by those values on five specific morphisms which correspond to the five natural transformations appearing in Cockett and Cruttwell’s definition:

- corresponding to the unique map $\epsilon : W \rightarrow \mathbb{N}$ is the *projection*,

$$p_T : T \rightarrow I;$$

- corresponding to the unit map $\eta : \mathbb{N} \rightarrow W$ is the *zero section*

$$0_T : I \rightarrow T;$$

- corresponding to the map

$$\phi : \mathbb{N}[x, y]/(x^2, xy, y^2) \rightarrow \mathbb{N}[z]/(z^2); \quad x \mapsto z, \quad y \mapsto z,$$

is the *addition*

$$+_T : T_2 \rightarrow T;$$

- corresponding to the symmetry map

$$\sigma : \mathbb{N}[x, y]/(x^2, y^2) \rightarrow \mathbb{N}[x, y]/(x^2, y^2); \quad x \mapsto y, \quad y \mapsto x,$$

is the *flip*

$$c_T : T^2 \rightarrow T^2;$$

- corresponding to the map

$$\delta : \mathbb{N}[z]/(z^2) \rightarrow \mathbb{N}[x, y]/(x^2, y^2); \quad z \mapsto xy,$$

is the *vertical lift*

$$\ell_T : T \rightarrow T^2.$$

Finally, the requirement that a tangent structure $T : \text{Weil}_1 \rightarrow \text{End}(\mathbb{X})$ preserve the vertical lift pullback (1.8) corresponds to the condition that Cockett and Cruttwell refer to in [CC17, 2.1] as the ‘Universality of the Vertical Lift’, i.e. that for all $M \in \mathbb{X}$ there is a pullback square (in \mathbb{X}) of the form

$$(1.13) \quad \begin{array}{ccc} TM \times_M TM & \longrightarrow & T(TM) \\ \downarrow & & \downarrow T(p) \\ M & \xrightarrow{0} & TM. \end{array}$$

EXAMPLES 1.14. Here are some of the standard examples of tangent categories. More examples, and more details, appear in the papers [CC14, CC18].

- (1) Let $\mathbb{X} = \mathbf{Mfld}$, the category of finite-dimensional smooth manifolds and smooth maps, and let $T : \mathbf{Mfld} \rightarrow \mathbf{Mfld}$ be the ordinary tangent bundle functor. Then there is a tangent structure on \mathbf{Mfld} with tangent bundle functor T and projection map given by the usual bundle projections $TM \rightarrow M$. The zero and addition maps come from the vector bundle structure on TM , and the flip and vertical lift can be defined directly in terms of tangent vectors [CC18, 2.2(i)].
- (2) The category \mathbf{Sch} of schemes has a tangent structure in which the tangent bundle functor $T : \mathbf{Sch} \rightarrow \mathbf{Sch}$ on a scheme X is the vector bundle associated to the \mathcal{O}_X -module of Kähler differentials of X (over $\mathrm{Spec} \mathbb{Z}$):

$$T(X) = \mathrm{Spec} \mathrm{Sym} \Omega_{X/\mathbb{Z}}.$$

See [Gar18, Ex. 2(iii)] for further details.

- (3) The category \mathbf{CRing} of commutative rings (with identity) has a tangent structure given by $T^A(R) := A \otimes R$, i.e. with tangent bundle functor $T(R) = R[x]/(x^2)$.
- (4) The category \mathbb{X} of ‘infinitesimally linear’ objects in a model of synthetic differential geometry (SDG) has a tangent structure whose tangent bundle functor $T : \mathbb{X} \rightarrow \mathbb{X}$ is given by the exponential $T(C) = C^D$ where D is an ‘object of infinitesimals’ in \mathbb{X} . See [CC14, 5.1] for more details.
- (5) Let \mathbb{X} be any category. Then there is a trivial tangent structure on \mathbb{X} in which $T^A : \mathbb{X} \rightarrow \mathbb{X}$ is the identity functor for every Weil-algebra A .
- (6) Let \mathbb{X} be a tangent category with tangent bundle functor T , and let \mathbb{J} be a category. Then the functor category $\mathrm{Fun}(\mathbb{J}, \mathbb{X})$ has a tangent structure whose tangent bundle functor is composition with T .

A Monoidal ∞ -Category of Weil-Algebras

As we have just seen, a fundamental role in the theory of ordinary tangent categories is played by the monoidal category $\mathbb{W}eil_1$ of Weil-algebras. There is a monoidal ∞ -category, which we denote $\mathbb{W}eil$, that plays the same role in the theory of tangent ∞ -categories. The goal of this chapter is to define $\mathbb{W}eil$, and to highlight the relationship between it and the 1-category $\mathbb{W}eil_1$.

Throughout this paper we use the quasi-categorical model for ∞ -categories, introduced by Boardman and Vogt [BV73], further developed by Joyal [Joy08], and popularized by Lurie [Lur09a]. A *quasi-category* or, for us, an *∞ -category* is a simplicial set \mathbb{X} which satisfies the ‘inner horn condition’ of [Lur09a, 1.1.2.4], which says that any ‘inner horn’ in \mathbb{X} , i.e. map of simplicial sets of the form

$$\Lambda_k^n \rightarrow \mathbb{X}$$

for $0 < k < n$, can be extended to an n -simplex $\Delta^n \rightarrow \mathbb{X}$. The nerve of a category is an ∞ -category, and the nerve construction determines a fully faithful embedding of the category of (small) categories into that of ∞ -categories. The image of that embedding consists of those ∞ -categories in which each inner horn has a *unique* filler. We will usually not distinguish between a category and its nerve. For example, when we say that an ∞ -category *is* a category, we mean it is the nerve of a category.

To motivate our construction of the ∞ -category $\mathbb{W}eil$, recall that a tangent structure on the category \mathbb{X} makes each projection map $p_M : T(M) \rightarrow M$ into a commutative monoid in the slice category $\mathbb{X}/_M$. Cockett and Cruttwell refer to this structure as an ‘additive bundle’ over M . The monoid structure is determined by the action of those Weil-algebras of the form W^n , for $n \geq 0$, and the fact that the tangent structure preserves the foundational pullbacks (1.7).

In a tangent ∞ -category those commutative monoids are replaced by E_∞ -structures, i.e. monoids that are associative and commutative up to higher coherent homotopies. Transitioning from the 1-category $\mathbb{W}eil_1$ to the ∞ -category $\mathbb{W}eil$ builds higher homotopies into the tangent structure, so that each projection map p_M is an E_∞ -monoid in the slice ∞ -category $\mathbb{X}/_M$.

Cranch describes in [Cra10] an ∞ -category (which we call \mathbb{E}_∞ , but therein was denoted Span) which plays the role of a Lawvere theory for E_∞ -monoids, in the sense that E_∞ -monoids in an ∞ -category \mathbb{X} correspond to product-preserving functors $\mathbb{E}_\infty \rightarrow \mathbb{X}$. Lawvere theories for ∞ -categories are developed in more detail by Berman; see [Ber20, 2.3] for further discussion of this example and others, which are related. The ∞ -category \mathbb{E}_∞ is also the ‘effective Burnside ∞ -category’ (of the category of finite sets) introduced by Barwick [Bar17].

Our construction of $\mathbb{W}eil$ will be closely based on that of \mathbb{E}_∞ , and so we take some time now to recall the definition of \mathbb{E}_∞ and how it encodes the structure of

an E_∞ -monoid. As Cranch’s terminology indicates, the ∞ -category \mathbb{E}_∞ can be described in terms of ‘spans’. The objects of \mathbb{E}_∞ are finite sets, and the morphisms from J to J' are diagrams of finite sets of the form

$$(2.1) \quad \begin{array}{ccc} & K & \\ s \swarrow & & \searrow t \\ J & & J'. \end{array}$$

To see the connection between spans of finite sets and monoids, consider the following alternative perspective on the objects and morphisms in \mathbb{E}_∞ . An object, i.e. a finite set J , is a basis for a free commutative monoid, which we denote \mathbb{N}^J . (The elements of \mathbb{N}^J are finite formal sums of elements of J .) A morphism in \mathbb{E}_∞ of the form (2.1) determines a monoid homomorphism $\phi : \mathbb{N}^J \rightarrow \mathbb{N}^{J'}$ as follows. For the basis element $j \in J$, we set

$$\phi(j) = \sum_{k \in s^{-1}(j)} t(k),$$

and then extend linearly to all of \mathbb{N}^J . When $s^{-1}(j) = \emptyset$, we have $\phi(j) = 0$.

Since any homomorphism ϕ is uniquely determined by its values on basis elements, and each $\phi(j)$ can be written as a finite sum of basis elements in $\mathbb{N}^{J'}$, any homomorphism can be represented by a span in a way that is unique up to isomorphism. A specific span is given by choosing *labels* for the basis elements appearing in all of the expressions $\phi(j)$, and taking the set K in (2.1) to be the set of all such labels. See [Cra10, Sec. 4.1] for more discussion of this process.

From this point of view, a morphism in the ∞ -category \mathbb{E}_∞ can be viewed as a monoid homomorphism $\phi : \mathbb{N}^J \rightarrow \mathbb{N}^{J'}$ together with a *labelling* of the terms in each of the expressions $\phi(j)$. Higher simplexes in \mathbb{E}_∞ build in permutations of those labels; in fact \mathbb{E}_∞ is the nerve of a bicategory whose 1-morphisms are these ‘labelled’ monoid homomorphisms, and whose 2-morphisms are bijections between the sets of labels which commute with the projection maps s and t . The additional flexibility described by those higher morphisms corresponds to the up-to-homotopy associativity and commutativity of an E_∞ -monoid.

Returning now to Weil-algebras, we define the ∞ -category $\mathbb{W}eil$ by adding ‘labels’ to each Weil-algebra morphism $\phi : A \rightarrow A'$ in the same way that they appear in the description of \mathbb{E}_∞ given above. Recall from Remark 1.5 that a basis for the additive structure on A is given by the finite set M_A of nonzero monomials in A . Thus a labelling of ϕ should consist of a span of finite sets of the form

$$(2.2) \quad \begin{array}{ccc} & K & \\ s \swarrow & & \searrow t \\ M_A & & M_{A'}. \end{array}$$

such that for each monomial $x_\alpha \in M_A$

$$\phi(x_\alpha) = \sum_{k \in s^{-1}(x_\alpha)} t(k).$$

However, there is additional structure on the sets in the diagram (2.2) which reflects the fact that A and A' also have a multiplicative structure, and that ϕ preserves that structure. In particular, the sets M_A and $M_{A'}$ each have the structure of a

partial commutative monoid, i.e. a commutative monoid in which the multiplication operation is only defined for *some* pairs of elements. The product of two elements of M_A is defined if and only if that product is nonzero in A , in which case it is another element of M_A . For example, if $A = W = \mathbb{N}[x]/(x^2)$, then we have $M_W = \{1, x\}$, and the products $1 \cdot 1$ and $1 \cdot x$ are defined, but $x \cdot x$ is not.

The set K in (2.2) provides labels for the monomials in A' which appear in the expressions $\phi(x_\alpha)$, where $x_\alpha \in M_A$. Suppose $x_\alpha, x_\beta \in M_A$ are such that the product $x_\alpha x_\beta$ is defined in M_A . Then each term in $\phi(x_\alpha x_\beta) = \phi(x_\alpha)\phi(x_\beta)$ can be identified with a pair of terms, one in $\phi(x_\alpha)$ and one in $\phi(x_\beta)$. However, if the product of a pair of terms is zero in A' , then it does not appear in the expression for $\phi(x_\alpha x_\beta)$. Thus K also has the structure of a partial commutative monoid and in such a way that the maps s and t preserve the multiplication. In other words, the diagram (2.2) lives in a category of (finite) partial commutative monoids, which for clarity we now define; for example, see [Weh17, Def. 2.1.1].

DEFINITION 2.3. A *finite partial commutative monoid* consists of a finite set B and a partial function $\bullet : B \times B \dashrightarrow B$ with the following properties:

- (1) for all $x, y \in B$, $x \cdot y$ is defined if and only if $y \cdot x$ is defined, in which case

$$x \cdot y = y \cdot x;$$

- (2) there is an element $1 \in B$ such that for all $x \in B$, $1 \cdot x$ is defined, and

$$1 \cdot x = x;$$

- (3) for all $x, y, z \in B$, $(x \cdot y) \cdot z$ is defined if and only if $x \cdot (y \cdot z)$ is defined, in which case

$$(x \cdot y) \cdot z = x \cdot (y \cdot z),$$

and we write this element as xyz . (In particular, when an expression of the form $y_1 \cdots y_r$ is defined in B , the product of any subset of these elements is also defined.)

For finite partial commutative monoids B, B' , a *homomorphism* from B to B' is a function $f : B \rightarrow B'$ such that

- (1) $f(1) = 1$;

- (2) for all $x, y \in B$, if $x \cdot y$ is defined, then $f(x) \cdot f(y)$ is defined, and

$$f(x \cdot y) = f(x) \cdot f(y).$$

Let FPCM denote the category of finite partial commutative monoids and their homomorphisms, with identities and composition given by the ordinary identities and composition of functions.

We can now give our definition of a labelled Weil-algebra morphism.

DEFINITION 2.4. A *labelled morphism* between Weil-algebras A and A' is a diagram in FPCM of the form

$$\begin{array}{ccc} & K & \\ s \swarrow & & \searrow t \\ M_A & & M_{A'} \end{array}$$

with the following properties:

- (1) every element of K is a unique (possibly empty, in which case it is the identity element 1) product of elements in the subset $K_1 := s^{-1}(\{x_1, \dots, x_n\})$, where x_1, \dots, x_n are the generators of the Weil-algebra A : see Remark 1.5;
- (2) for $k, k' \in K$, the product $k \cdot k'$ is defined in K if and only if the product $t(k) \cdot t(k')$ is defined in $M_{A'}$, i.e. is a nonzero monomial in A' .

The *underlying Weil-algebra morphism* of a labelled morphism is the function $\phi : A \rightarrow A'$ given on monomials $x_\alpha \in M_A$ by

$$(2.5) \quad \phi(x_\alpha) := \sum_{k \in s^{-1}(x_\alpha)} t(k)$$

and extended linearly to A . Lemma 2.10 below shows that ϕ is indeed an algebra morphism. We say that K is a *set of labels* for ϕ .

REMARK 2.6. The motivation for part (1) of Definition 2.4 is that for a Weil-algebra morphism $\phi : A \rightarrow A'$ each term in $\phi(x_{i_1} \cdots x_{i_r})$ can be uniquely identified with a product of terms in the expressions $\phi(x_{i_j})$. For (2), the rationale is that if the product of a term in $\phi(x_\alpha)$ with a term in $\phi(x_\beta)$ is zero in A' , then that product does not appear in the expression for $\phi(x_\alpha x_\beta)$.

EXAMPLE 2.7. Consider the Weil-algebra morphism $\phi : W \otimes W \rightarrow W \otimes W$ given by

$$\phi(x) = xy + y, \quad \phi(y) = 2x.$$

An example of a labelling for ϕ is a span of the form

$$\begin{array}{ccc} & \{1, a, b, c, d, bc, bd\} & \\ s \swarrow & & \searrow t \\ \{1, x, y, xy\} & & \{1, x, y, xy\} \end{array}$$

where $s(a) = s(b) = x$, $s(c) = s(d) = y$, and $t(a) = xy$, $t(b) = y$, $t(c) = t(d) = x$, and both s and t are extended multiplicatively.

EXAMPLE 2.8. For any morphism of Weil-algebras $\phi : A \rightarrow A'$ in which each generator of A is mapped to either a single monomial of A' or to zero, there is a canonical labelling given by a diagram of the form

$$\begin{array}{ccc} & K & \\ \swarrow & & \searrow \phi|_K \\ M_A & & M_{A'} \end{array}$$

where $K \hookrightarrow M_A$ is the inclusion of the subset consisting of those monomials which do not map to 0 in A' . For example, the identity morphism on the Weil-algebra A has a canonical labelling given by the span consisting of two copies of the identity function on M_A .

REMARK 2.9. One consequence of condition (1) in Definition 2.4 is that the only element of K for which $s(k) = 1$ is $k = 1$: every other element in K is a product of at least one element in K_1 and so is mapped by s to a monomial in A of degree at least 1.

Similarly, a consequence of (2) is that the only element of K for which $t(k) = 1$ is $k = 1$: suppose $t(k) = 1$; then $t(k) \cdot t(k) = 1 \cdot 1 = 1$ is nonzero in A' , and so by

(2), $k \cdot k$ is defined in K . Then $s(k)s(k) = s(k \cdot k)$ is an element of M_A , but the only nonzero square monomial in the Weil-algebra A is 1, so $s(k) = 1$ and hence $k = 1$ as already noted.

LEMMA 2.10. *The map $\phi : A \rightarrow A'$ defined in (2.5) is a Weil-algebra morphism.*

PROOF. Since we have defined ϕ on an additive basis and extended linearly, ϕ evidently preserves the additive structure. It remains to show that ϕ also preserves the multiplicative structure. Firstly, we have

$$\phi(1) = \sum_{k \in s^{-1}(1)} t(k) = t(1) = 1.$$

To show that ϕ is multiplicative, it is sufficient to show that $\phi(x_\alpha x_\beta) = \phi(x_\alpha)\phi(x_\beta)$ for any two nonzero monomials x_α, x_β in A . We split the argument into two cases, depending on whether the product $x_\alpha x_\beta$ is zero or not in A .

Suppose $x_\alpha x_\beta$ is not zero in A , and hence is an element of M_A . We will establish a one-to-one correspondence between the set $s^{-1}(x_\alpha x_\beta)$ and the set of pairs $(k', k'') \in s^{-1}(x_\alpha) \times s^{-1}(x_\beta)$ such that $t(k')t(k'')$ is nonzero in A' . (Note that if $\phi(x_\alpha x_\beta) = 0$ in A' , then our argument will imply that both of these sets are empty.)

Take first an element $k \in K$ such that $s(k) = x_\alpha x_\beta$. We know $k = k_1 \cdots k_r$ for some unique $k_1, \dots, k_r \in K_1$. The generators of the Weil-algebra A which appear in the nonzero monomial $x_\alpha x_\beta$ are all distinct, so let k' be the product of all those k_i for which $s(k_i)$ is in x_α , and let k'' be the product of all those k_j for which $s(k_j)$ is in x_β . Those products exist in K because the larger product defining k itself exists. So we have a uniquely determined pair $(k', k'') \in s^{-1}(x_\alpha) \times s^{-1}(x_\beta)$ and $t(k')t(k'') = t(k)$ is a nonzero monomial in A' .

Conversely, suppose (k', k'') is such a pair. Then the product $t(k')t(k'')$ is defined in $M_{A'}$, and so the product $k = k'k''$ is defined in K , and satisfies $s(k) = s(k')s(k'') = x_\alpha x_\beta$. This construction establishes the one-to-one correspondence. We therefore have

$$\phi(x_\alpha x_\beta) = \sum_{s(k)=x_\alpha x_\beta} t(k) = \sum_{s(k')=x_\alpha, s(k'')=x_\beta} t(k')t(k'') = \phi(x_\alpha)\phi(x_\beta).$$

For the second case, suppose that $x_\alpha x_\beta = 0$ in A . Our definition of ϕ automatically preserves the additive structure, so we then have $\phi(x_\alpha x_\beta) = 0$. Suppose $\phi(x_\alpha)\phi(x_\beta)$ is not zero. Then it includes a nonzero term $t(k')t(k'')$ for some $k' \in s^{-1}(x_\alpha)$ and $k'' \in s^{-1}(x_\beta)$. It follows that the product $k'k''$ is defined in K , and then $s(k'k'') = s(k')s(k'') = x_\alpha x_\beta$, which is a contradiction since the product $x_\alpha x_\beta$ is not defined in M_A . So $\phi(x_\alpha)\phi(x_\beta) = 0 = \phi(x_\alpha x_\beta)$.

This completes the check that ϕ preserves the multiplicative structure, and so $\phi : A \rightarrow A'$ is a Weil-algebra morphism. \square

The ∞ -category Weil has as its objects the Weil-algebras and as its 1-morphisms the labelled Weil-algebra morphisms of Definition 2.4. As in Cranch's definition of the ∞ -category \mathbb{E}_∞ (see [Cra10, Def. 4.4] where it is denoted $\text{Span}(\text{FinSet})$), the 2-simplexes in Weil are certain larger diagrams in the category FPCM .

DEFINITION 2.11. A 2-simplex in Weil is a diagram of partial commutative monoids of the form

$$\begin{array}{ccccc}
 & & K'' & & \\
 & & \swarrow s'' & \searrow t'' & \\
 & K & & & K' \\
 & \swarrow s & & \swarrow s' & \searrow t' \\
 M_A & & M_{A'} & & M_{A''}
 \end{array}$$

such that

- (1) each of the pairs of functions (s, t) , (s', t') , and $(s \circ s'', t' \circ t'')$ is a labelled Weil-algebra morphism in the sense of Definition 2.4;
- (2) the central square of this diagram is a pullback in FPCM.

REMARK 2.12. Pullbacks in FPCM are created in the underlying category of finite sets. Explicitly, the partial monoid structure on a pullback $B' \times_B B''$ is such that for elements (b'_1, b'_1) and (b'_2, b'_2) of the pullback, the product is defined if and only if $b'_1 b'_2$ and $b''_1 b''_2$ are both defined, in which case

$$(b'_1, b'_1) \cdot (b'_2, b'_2) = (b'_1 \cdot b'_2, b''_1 \cdot b''_2).$$

REMARK 2.13. The 2-simplexes in an ∞ -category encode composition in the sense that each 2-simplex exhibits one of its edges as a ‘composite’ of two others, though this composite is not in general unique. In the case of Definition 2.11, the labelled Weil-algebra morphism with labels in K'' can be viewed as a composite of the labelled Weil-algebra morphisms with labels K and K' . We now show that on the underlying morphisms this composition corresponds to ordinary composition of functions.

LEMMA 2.14. *For a 2-simplex in Weil as in Definition 2.11, the underlying Weil-algebra morphism from A to A'' is the composite of the underlying morphisms from A to A' and A' to A'' .*

PROOF. This is a straightforward calculation. If $\phi : A \rightarrow A'$, $\phi' : A' \rightarrow A''$, $\phi'' : A \rightarrow A''$ are the underlying morphisms of the three labelled morphisms in the definition, then

$$\phi''(x_\alpha) = \sum_{s(s''(k''))=x_\alpha} t'(t''(k'')).$$

From the pullback condition (2), we can identify each such k'' with a pair (k, k') where $s(k) = x_\alpha$, and $t(k) = s'(k')$. Then we have

$$\phi''(x_\alpha) = \sum_{s(k)=x_\alpha} \sum_{s'(k')=t(k)} t'(k') = \sum_{s(k)=x_\alpha} \phi'(t(k)) = \phi' \left(\sum_{s(k)=x_\alpha} t(k) \right) = \phi'(\phi(x_\alpha)).$$

□

The higher-degree simplexes in Weil follow a similar pattern, which we now describe.

DEFINITION 2.15. For each non-negative integer n , let J_n be the poset of nonempty intervals in the totally ordered set $[n] = \{0, 1, \dots, n\}$, ordered by reverse inclusion. That is, an element of J_n is a set of the form

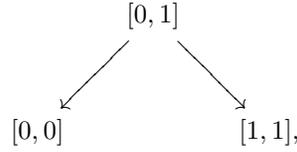
$$[i, j] = \{i, i + 1, \dots, j - 1, j\}$$

for some $0 \leq i \leq j \leq n$. An order-preserving function $f : [n] \rightarrow [n']$ induces a functor $J_n \rightarrow J_{n'}$ by

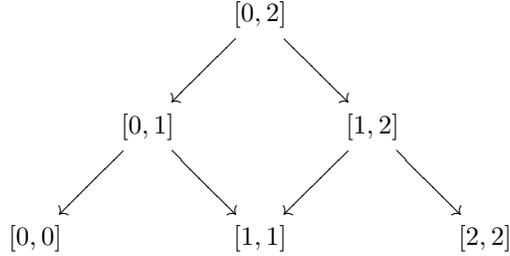
$$f_* : [i, j] \mapsto [f(i), f(j)].$$

These functors make the sequence $(J_n \mid n \geq 0)$ into a cosimplicial object in the category of posets.

EXAMPLE 2.16. The poset J_0 has the single element $[0, 0]$, J_1 is the indexing category for a span diagram:



and the poset J_2 is the indexing diagram



for the 2-simplexes in $\mathbb{W}eil$.

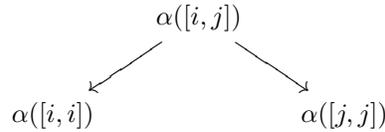
DEFINITION 2.17. Let $\mathbb{W}eil$ be the simplicial set whose n -simplexes are those J_n -indexed diagrams of finite partial commutative monoids, that is, those functors $\alpha : J_n \rightarrow \mathbf{FPCM}$, with the following properties:

- (0) for each $i \in [n]$:

$$\alpha([i, i]) = M_{A_i}$$

for some Weil-algebra A_i ;

- (1) for each $i < j$ in $[n]$: the diagram



is a labelled Weil-algebra morphism from A_i to A_j , as described in Definition 2.4;

(2) for each $i < i' \leq j' < j$ in $[n]$, the square diagram

$$\begin{array}{ccc}
 & \alpha([i, j]) & \\
 \swarrow & & \searrow \\
 \alpha([i, j']) & & \alpha([i', j]) \\
 \searrow & & \swarrow \\
 & \alpha([i', j']) &
 \end{array}$$

is a pullback in FPCM , i.e. a pullback of finite sets.

The simplicial structure maps in Weil are given by precomposition with the functors labelled f_* in Definition 2.15. If α satisfies conditions (0)-(2), then so does the diagram αf_* , so Weil is a simplicial set.

Cranch shows in [Cra10, Prop. 4.5] that \mathbb{E}_∞ is an ∞ -category by identifying it as the nerve of a bicategory of spans of finite sets in which all 2-morphisms are invertible. A similar argument is true for Weil . In fact, using Cranch's work, we can identify Weil with the nerve of a sub-bicategory of the bicategory $2\text{Span}(\text{FPCM})$ as defined in [Cra10, Sec. 4.2]. For clarity, we spell out the details of this bicategory here. We follow the definition of bicategory given by Duskin in [Dus01, 5.1].

PROPOSITION 2.18. *The simplicial set Weil is an ∞ -category.*

PROOF. We will define a bicategory with objects the Weil-algebras, 1-cells the labelled Weil-algebra morphisms, and 2-cells the commutative diagrams in FPCM of the form

$$\begin{array}{ccc}
 & K & \\
 \swarrow s & \downarrow \alpha & \searrow t \\
 M_A & \cong & M_{A'} \\
 \swarrow s' & \downarrow & \searrow t' \\
 & K' &
 \end{array}$$

where the map α is an isomorphism in FPCM (i.e. a bijection which preserves multiplication). Composition and identities for 2-cells are given by the composition and identities of these underlying isomorphisms. Since these 2-cells are all invertible, the nerve of this bicategory is an ∞ -category by [Dus01, 8.6].

Horizontal composition of 1-cells is given as follows. Suppose $\phi : A \rightarrow A'$ and $\phi' : A' \rightarrow A''$ are labelled Weil-algebra morphisms with sets of labels K and K' . We then form the diagram of partial commutative monoids

$$\begin{array}{ccccc}
 & & K'' & & \\
 & & \swarrow \cdots & \searrow \cdots & \\
 & K & & K' & \\
 \swarrow s & \searrow t & & \swarrow s' & \searrow t' \\
 M_A & & M_{A'} & & M_{A''}
 \end{array}$$

where K'' is a pullback of the central square in the category FPCM. As described in Remark 2.12, we can take the underlying set of K'' to be the standard pullback of the sets K and K' over $M_{A'}$. The product in K'' of two elements (k_1, k'_1) and (k_2, k'_2) is defined and equal to $(k_1 k_2, k'_1 k'_2)$ if and only if both of those products are defined in K and K' respectively.

The main substance of this proof now is to show that K'' is the vertex of a labelled Weil-algebra morphism from A to A'' by verifying conditions (1) and (2) in Definition 2.4. That morphism is the composite of the 1-cells ϕ and ϕ' .

For condition (1), take an arbitrary element (k, k') of K'' . Then we have $k = k_1 \cdots k_r$ for some elements $k_1, \dots, k_r \in K_1 = s^{-1}(\{x_1, \dots, x_n\})$, and $k' = k'_1 \cdots k'_q$ for $k'_1, \dots, k'_q \in (s')^{-1}(\{x_1, \dots, x_{n'}\})$. Then

$$t(k_1) \cdots t(k_r) = t(k) = s'(k') = s'(k'_1) \cdots s'(k'_q).$$

Since each $s'(k'_j)$ is a single generator of A' , we must have, for each $i = 1, \dots, r$:

$$t(k_i) = s'(k'_{i_1}) \cdots s'(k'_{i_p})$$

for some sequence of indexes i_1, \dots, i_p . Since no generator in the nonzero monomial $s'(k')$ can be repeated, that sequence is uniquely determined for each i . If we write $\ell'_i = k'_{i_1} \cdots k'_{i_p}$, then (k_i, ℓ'_i) is an element of K'' with the property that $s(s''(k_i, \ell'_i)) = s(k_i) \in \{x_1, \dots, x_n\}$. We therefore have

$$(k, k') = (k_1, \ell'_1) \cdots (k_r, \ell'_r)$$

in K'' . Since the decomposition $k = k_1 \cdots k_r$ is unique, the same is true for (k, k') , and so we have shown that every element of K'' is a unique product of elements in $K''_1 = (s \circ s'')^{-1}(\{x_1, \dots, x_n\})$.

For condition (2), now take (k_1, k'_1) and (k_2, k'_2) in K'' , and suppose that $(t' \circ t'')(k_1, k'_1) \cdot (t' \circ t'')(k_2, k'_2)$ is a nonzero monomial in A'' . That means $t'(k'_1)t'(k'_2)$ is nonzero, and so the product $k'_1 k'_2$ is defined in K' . We then have

$$t(k_1)t(k_2) = s'(k'_1)s'(k'_2) = s'(k'_1 k'_2).$$

So $t(k_1)t(k_2)$ is nonzero in A' , and therefore the product $k_1 k_2$ is defined in K . It follows that the product $(k_1, k'_1) \cdot (k_2, k'_2)$ is defined in K'' , and is equal to $(k_1 k_2, k'_1 k'_2)$.

The horizontal composition for our desired bicategory is given on 1-cells by forming the pullback in FPCM as described above. For a pair of 2-cells as in the following diagram:

$$\begin{array}{ccccc} & & K_0 & & K_1 \\ & \swarrow & \downarrow & \searrow & \swarrow \\ M_{A_0} & & \alpha_0 \cong & & M_{A_1} & & \cong & & M_{A_2} \\ & \swarrow & \downarrow & \searrow & \swarrow & & \downarrow & \searrow & \\ & & K'_0 & & K'_1 & & & & \end{array}$$

the horizontal composite is the map on pullbacks $K_0 \times_{M_{A_1}} K_1 \rightarrow K'_0 \times_{M_{A_1}} K'_1$ induced by the pair (α_0, α_1) .

The pseudo-identity 1-cell on a Weil-algebra A is given by the identity morphism 1_A with its canonical labelling in the sense of Example 2.8, that is the span consisting of two copies of the identity homomorphism on M_A .

The pseudo-identity isomorphisms for a 1-cell with vertex K are the standard isomorphisms $K \rightarrow M_A \times_{M_A} K$ and $K \rightarrow K \times_{M_A} M_A$. The associativity isomorphism for three composable 1-cells is the standard isomorphism between pullbacks of the form

$$K_0 \times_{M_{A_1}} (K_1 \times_{M_{A_2}} K_2) \cong (K_0 \times_{M_{A_1}} K_1) \times_{M_{A_2}} K_2.$$

Uniqueness of these induced isomorphisms between pullbacks implies all the required compatibilities between these data, yielding the desired bicategory.

Finally, the identification of the nerve of this bicategory with the simplicial set $\mathbb{W}\text{eil}$ follows from [Cra10, Prop. 4.5]. Thus $\mathbb{W}\text{eil}$ is an ∞ -category. \square

The connection between $\mathbb{W}\text{eil}$ and $\mathbb{W}\text{eil}_1$ is given by the following lemma.

LEMMA 2.19. *The homotopy category of the ∞ -category $\mathbb{W}\text{eil}$ is isomorphic to $\mathbb{W}\text{eil}_1$.*

PROOF. The homotopy category has the same set of objects as the ∞ -category. Morphisms in the homotopy category are isomorphism classes of 1-morphisms in the bicategory constructed in the proof of Proposition 2.18. Since any two isomorphic labelled Weil-algebra morphisms have the same underlying morphism, and any two labellings of a given morphism are isomorphic, those isomorphism classes can be identified with the underlying (unlabelled) Weil-algebra morphisms. It follows from Lemma 2.14 that composition in the homotopy category of $\mathbb{W}\text{eil}$ agrees with composition in $\mathbb{W}\text{eil}_1$, and so the categories are isomorphic. \square

The Monoidal Structure on $\mathbb{W}\text{eil}$. As we saw in Chapter 1, the monoidal structure on the category of Weil-algebras given by tensor product plays an essential role in the definition of tangent category. In order to describe the corresponding structure on the ∞ -category $\mathbb{W}\text{eil}$, we need to introduce a model for monoidal ∞ -categories.

One of the appealing aspects of quasi-categories as a model for ∞ -categories from our perspective is that they admit a robust theory of monoidal structures. In full generality, that theory is typically based on Lurie’s ∞ -operads [Lur17, 2.1.2.13], which are complicated to implement. Fortunately, the ∞ -category $\mathbb{W}\text{eil}$ admits a *strict* monoidal structure which will be much easier for us to work with.

DEFINITION 2.20. A *strict monoidal ∞ -category* is a simplicial monoid \mathbb{M}^\otimes for which the underlying simplicial set \mathbb{M} is an ∞ -category. We will often drop the superscript \otimes and simply refer to ‘the strict monoidal ∞ -category \mathbb{M} ’ with the monoid structure understood.

REMARK 2.21. We use the phrase ‘strict monoidal’ to emphasize that these monoidal structures are strictly associative and unital, in contrast to other models for monoidal structures on an ∞ -category, such as that given by Lurie’s ∞ -operads [Lur17, 2.1.2.13], in which non-trivial associativity isomorphisms (and higher coherences) are built in to the definition. Moreover, as for ordinary monoidal categories, every monoidal ∞ -category is, in a sense, equivalent to one that is strict; see [Lur17, 4.1.8.7] for an explanation of this claim.

EXAMPLE 2.22. For any ∞ -category \mathbb{X} , the functor ∞ -category

$$\text{End}(\mathbb{X}) := \text{Fun}(\mathbb{X}, \mathbb{X}),$$

given by the ordinary simplicial mapping space for the simplicial set \mathbb{X} , is a strict monoidal ∞ -category, which we denote $\text{End}(\mathbb{X})^\circ$, with monoidal structure given by composition of functors.

We now introduce the structure of a simplicial monoid on the ∞ -category $\mathbb{W}\text{eil}$ which, on objects, is given by the tensor product of Weil-algebras. To understand that structure, recall that the Weil-algebra A is represented as an object in $\mathbb{W}\text{eil}$ by the partial commutative monoid M_A of nonzero monomials in A . Now notice that the nonzero monomials in $A \otimes A'$ are each uniquely given by the tensor of a nonzero monomial in A with a nonzero monomial in A' . In other words, there is an isomorphism (of partial commutative monoids):

$$M_{A \otimes A'} \cong M_A \times M_{A'}.$$

Our plan is therefore to model the tensor product of Weil-algebras with the cartesian product of partial commutative monoids.

One technical wrinkle in carrying out that plan is that the usual model for the cartesian product in FPCM (based on the cartesian product of underlying sets) is not strictly associative or unital. For example, the trivial monoid structure on any one-element set is a unit up to isomorphism, but there is no object that is a *strict* unit. In order to construct the desired strict monoidal structure on $\mathbb{W}\text{eil}$, we therefore need to introduce a different model for that product. We use a construction due to Schauenburg [Sch01].

PROPOSITION 2.23. *There is a strict monoidal category $(\text{FPCM}, \times, \{1\})$, with unit object given by the one-element set $\{1\}$ with its unique partial monoid structure, such that $B \times C$ is naturally a categorical product of B and C for any $B, C \in \text{FPCM}$. Moreover, any object K of FPCM has a unique expression of the form*

$$K = K_1 \times \cdots \times K_r$$

where K_1, \dots, K_r is a (possibly empty) sequence of ‘indecomposable’ objects, i.e. those which are not equal to a product $K' \times K''$ unless either K' or K'' equals the unit object $\{1\}$, and also not equal to $\{1\}$ themselves.

The construction of \times is essentially due to Schauenburg, who proves in [Sch01, 4.3] that any monoidal structure on a category of ‘structured’ sets is monoidally equivalent to a strict monoidal structure on the same category. Applying that argument to the usual cartesian product of sets with partial monoid structures described in Remark 2.12, we obtain a strict monoidal product on FPCM.

However, Schauenburg’s construction does not have the additional unique decomposition property, and so we will introduce a variant of that construction in order to prove Proposition 2.23. The details of this variant are irrelevant to the broader purpose of the paper, and therefore we postpone the proof until the end of this chapter; see Definition 2.37 and after.

REMARK 2.24. The underlying set of a partial commutative monoid of the form $B \times C$, where \times is as in Proposition 2.23, is not (precisely) equal to the set of ordered pairs (b, c) with $b \in B$ and $c \in C$. However, the universal property of the product implies that there is a canonical bijection between those sets. We can therefore refer to elements of $B \times C$ via the notation (b, c) , with the understanding that we mean the element of $B \times C$ which corresponds to that ordered pair under this canonical bijection.

Working with elements of $B \times C$ in this way, the partial monoid structure is given as in Remark 2.12: a product $(b, c) \cdot (b', c')$ is defined in $B \times C$ if and only if $b \cdot b'$ and $c \cdot c'$ are defined in B and C respectively, in which case we have

$$(b, c) \cdot (b', c') = (b \cdot b', c \cdot c').$$

EXAMPLE 2.25. In this paper, the most important objects in FPCM are the finite partial commutative monoids M_A given by the sets of nonzero monomials in a Weil-algebra A . We noticed above that for a Weil-algebra $A = W^{J_1} \otimes \cdots \otimes W^{J_r}$, there is an isomorphism

$$M_A \cong M_{W^{J_1}} \times \cdots \times M_{W^{J_r}}.$$

From now on we take this isomorphism as our *definition* of the finite partial commutative monoid M_A .

To be precise, for a nonempty finite set J , we define the finite partial commutative monoid:

$$M_{W^J} := J \sqcup \{1\}$$

with only those products involving the identity element, such as $j \cdot 1 = j$, defined in M_{W^J} . We then set

$$M_A := M_{W^{J_1}} \times \cdots \times M_{W^{J_r}}.$$

This choice means that in FPCM we now have an *equality*

$$M_{A \otimes A'} = M_A \times M_{A'}$$

for Weil-algebras A and A' , and $M_{\mathbb{N}}$ is *equal* to the unit object $\{1\}$ for the strict monoidal structure on FPCM.

We can now introduce a central object in this paper: the strict monoidal ∞ -category of Weil-algebras.

DEFINITION 2.26. Let $\mathbb{W}\text{eil}^{\otimes}$ be the following strict monoidal ∞ -category whose underlying simplicial set $\mathbb{W}\text{eil}$ is as in Definition 2.17. For two n -simplexes $\alpha, \beta : J_n \rightarrow \text{FPCM}$, we define $\alpha \otimes \beta : J_n \rightarrow \text{FPCM}$ on objects by

$$(\alpha \otimes \beta)([i, j]) := \alpha([i, j]) \times \beta([i, j]),$$

where \times denotes the strict monoidal product on FPCM described in Proposition 2.23, and on morphisms by the induced map between products. It follows from the next lemma that \otimes is a strict monoidal structure on the ∞ -category $\mathbb{W}\text{eil}$.

LEMMA 2.27. *Let α and β be n -simplexes in $\mathbb{W}\text{eil}$. Then $\alpha \otimes \beta$ is an n -simplex in $\mathbb{W}\text{eil}$.*

PROOF. We have to verify that the diagram in FPCM which underlies $\alpha \otimes \beta$ satisfies conditions (0)-(2) of Definition 2.17. Condition (0) is satisfied because, as described in Example 2.25, we have

$$(\alpha \otimes \beta)([i, i]) = \alpha([i, i]) \times \beta([i, i]) = M_{A_i} \times M_{A'_i} = M_{A_i \otimes A'_i}.$$

Condition (2) follows from the fact that the cartesian product of pullback diagrams is also a pullback.

For condition (1), we have to show that given labelled Weil-algebra morphisms

$$\begin{array}{ccc} & K & \\ s \swarrow & & \searrow t \\ M_{A_0} & & M_{A_1} \end{array} \quad \begin{array}{ccc} & K' & \\ s'' \swarrow & & \searrow t' \\ M_{A'_0} & & M_{A'_1} \end{array}$$

the product diagram

$$\begin{array}{ccc} & K \times K' & \\ (s, s') \swarrow & & \searrow (t, t') \\ M_{A_0 \otimes A'_0} & & M_{A_1 \otimes A'_1} \end{array}$$

is also a labelled Weil-algebra morphism, i.e. satisfies conditions (1)-(2) of Definition 2.4.

The generators of $M_{A_0 \otimes A'_0} = M_{A_0} \times M_{A'_0}$ are those of the form $(x_i, 1)$ and $(1, x_i)$ where x_i is a generator for A_0 or A'_0 . (Recall that we can label elements of $M_{A_0} \times M_{A'_0}$ by ordered pairs as described in Remark 2.24.) The objects mapped by the function (s, s') to one of those generators are precisely those of the form $(k, 1)$, where $k \in K_1$, or $(1, k')$, where $k' \in K'_1$. Thus these are the elements of $(K \times K')_1$.

For an arbitrary element $(k, k') \in K \times K'$, we have a unique decomposition $k = k_1 \cdots k_r$ and $k' = k'_1 \cdots k'_q$ where $k_i \in K_1$ and $k'_i \in K'_1$. Then

$$(k, k') = (k_1, 1) \cdots (k_r, 1) \cdot (1, k'_1) \cdots (1, k'_q)$$

is the unique decomposition of (k, k') into a product of elements in $(K \times K')_1$. This verifies condition (1) of 2.4.

Now suppose that $(t, t')(k_1, k'_1) \cdot (t, t')(k_2, k'_2)$ is defined in $M_{A_1 \otimes A'_1}$. That object is equal to

$$(t(k_1)t(k_2), t'(k'_1)t'(k'_2)),$$

and so $t(k_1)t(k_2)$ is defined in M_{A_1} , and $t'(k'_1)t'(k'_2)$ is defined in $M_{A'_1}$. It follows that $k_1 k_2$ is defined in K , and $k'_1 k'_2$ is defined in K' , and so

$$(k_1, k'_1) \cdot (k_2, k'_2) = (k_1 k_2, k'_1 k'_2)$$

is defined in $K \times K'$. That verifies condition (2), completing the proof that $\alpha \otimes \beta$ is an n -simplex in $\mathbb{W}eil$. \square

REMARK 2.28. The strict monoidal category $\mathbb{W}eil^\otimes$ induces a strict monoidal structure on the homotopy category $h\mathbb{W}eil \cong \mathbb{W}eil_1$, which agrees with the ordinary tensor product of Weil-algebras.

Some limits and colimits in $\mathbb{W}eil$. In this section we show that the tangent pullbacks in the category $\mathbb{W}eil_1$ extend to the ∞ -category $\mathbb{W}eil$, and that the tensor product of Definition 2.26 is still a coproduct in $\mathbb{W}eil$. To prove these facts it will be helpful to understand the spaces of morphisms between two objects in $\mathbb{W}eil$.

LEMMA 2.29. *Let A, A' be Weil-algebras. For each Weil-algebra morphism $\phi : A \rightarrow A'$, we can write*

$$\phi(x_i) = \sum_{y \in M_{A'}} n_{i,y} y$$

where x_i is one of the generators of A , y ranges over the distinct nonzero monomials in A' , that is, the elements of $M_{A'}$, and $n_{i,y}$ is a positive integer. Then the mapping space in the ∞ -category $\mathbb{W}eil$ is a disjoint union of components indexed by the Weil-algebra morphisms, each of which is a product of classifying spaces of symmetric groups:

$$\mathrm{Hom}_{\mathbb{W}eil}(A, A') \simeq \bigsqcup_{\phi: A \rightarrow A'} \prod_{i,y} B\Sigma_{n_{i,y}}.$$

PROOF. The components of $\mathrm{Hom}_{\mathbb{W}eil}(A, A')$ are the elements of the set of maps in the homotopy category $\mathbb{W}eil_1$, i.e. they are the Weil-algebra morphisms from A to A' . The component corresponding to the Weil-algebra morphism $\phi : A \rightarrow A'$ is the classifying space of the group of automorphisms of a labelling of ϕ in the bicategory introduced in the proof of Proposition 2.18. That group is the group of multiplicative bijections $\alpha : K \rightarrow K$ which commute with the maps $s : K \rightarrow M_A$ and $t : K \rightarrow M_{A'}$. Such a bijection α is determined by its values on the generating set K_1 , where it can freely permute the $n_{i,y}$ elements in the preimage $(s \times t)^{-1}(x_i, y)$. Therefore the automorphism group is the product of symmetric groups on each of these positive integers. \square

We can use Lemma 2.29 to show that \mathbb{N} is both initial and terminal in the ∞ -category $\mathbb{W}eil$.

LEMMA 2.30. *The ∞ -category $\mathbb{W}eil$ is pointed with null object \mathbb{N} .*

PROOF. Since \mathbb{N} is a null object in $\mathbb{W}eil_1$, the mapping spaces $\mathrm{Hom}_{\mathbb{W}eil}(A, \mathbb{N})$ and $\mathrm{Hom}_{\mathbb{W}eil}(\mathbb{N}, A)$ have a unique component. The unique morphism $\phi : A \rightarrow \mathbb{N}$ is given by $\phi(x_i) = 0$ for all generators x_i of A , and so by Lemma 2.29, $\mathrm{Hom}_{\mathbb{W}eil}(A, \mathbb{N}) \simeq *$. Since \mathbb{N} itself has zero generators, $\mathrm{Hom}_{\mathbb{W}eil}(\mathbb{N}, A)$ is also contractible. Therefore \mathbb{N} is both terminal and initial in $\mathbb{W}eil$. \square

Recall that a crucial role in the theory of ordinary tangent categories is played by certain pullback diagrams in the category $\mathbb{W}eil_1$ (Lemmas 1.7, 1.8). We now show that the corresponding diagrams in the ∞ -category $\mathbb{W}eil$ are also pullbacks.

PROPOSITION 2.31. *Each of the tangent pullback squares in Lemmas 1.7 and 1.8 is the underlying diagram of a pullback in the ∞ -category $\mathbb{W}eil$. We refer to these as the tangent pullbacks in $\mathbb{W}eil$.*

PROOF. We use the following criterion for a diagram in an ∞ -category $\mathbb{W}eil$ to be a pullback, which is a consequence of [Lur09a, 4.2.4.1]: that for any object A , applying the mapping space construction $\mathrm{Hom}_{\mathbb{W}eil}(A, -)$ yields a homotopy pullback square of spaces.

Now consider the foundational pullback diagram in Lemma 1.7. Each Weil-algebra morphism in that diagram has a canonical labelling described in Example 2.8. The fact that the diagram commutes in $\mathbb{W}eil_1$ implies that those canonical labellings can be uniquely extended to a commutative diagram in the ∞ -category $\mathbb{W}eil$ of the same form.

Now take the corresponding diagram of spaces given by applying the mapping space construction:

$$(2.32) \quad \begin{array}{ccc} \mathrm{Hom}_{\mathbb{W}\mathrm{eil}}(A, W^{J \sqcup J'}) & \longrightarrow & \mathrm{Hom}_{\mathbb{W}\mathrm{eil}}(A, W^J) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathbb{W}\mathrm{eil}}(A, W^{J'}) & \longrightarrow & \mathrm{Hom}_{\mathbb{W}\mathrm{eil}}(A, \mathbb{N}) \end{array}$$

Since the original diagram in 1.7 is a pullback in $\mathbb{W}\mathrm{eil}_1$, we know that the components of $\mathrm{Hom}_{\mathbb{W}\mathrm{eil}}(A, W^{J \sqcup J'})$ correspond to the components of the pullback of the remainder of the diagram. So fix such a component, i.e. a Weil-algebra morphism $\phi : A \rightarrow W^{J \sqcup J'}$.

Each monomial appearing as one of the summands in one of the expressions $\phi(x_i) \in W^{J \sqcup J'}$ is either an element of J or an element of J' . (The constant monomial 1 cannot appear because ϕ is a semi-ring homomorphism, and there are no non-trivial products in $M_{W^{J \sqcup J'}}$.) The corresponding expressions for the projected maps $A \rightarrow W^J$ and $A \rightarrow W^{J'}$ are given simply by deleting the terms in J' or J , respectively. For each i , we therefore have a (homotopy) pullback square

$$\begin{array}{ccc} \prod_{j \in J \sqcup J'} B\Sigma_{n_{i,j}} & \longrightarrow & \prod_{j \in J} B\Sigma_{n_{i,j}} \\ \downarrow & & \downarrow \\ \prod_{j \in J'} B\Sigma_{n_{i,j}} & \longrightarrow & * \end{array}$$

Taking the product over all generators x_i of A , and then the disjoint union over all Weil-algebra morphisms $\phi : A \rightarrow W^{J \sqcup J'}$, we obtain by Lemma 2.29 the desired homotopy pullback diagram (2.32).

Similarly, for the vertical lift diagram in Lemma 1.8, we want to show that the corresponding diagram of spaces

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{W}\mathrm{eil}}(A, W^2) & \longrightarrow & \mathrm{Hom}_{\mathbb{W}\mathrm{eil}}(A, W \otimes W) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathbb{W}\mathrm{eil}}(A, \mathbb{N}) & \longrightarrow & \mathrm{Hom}_{\mathbb{W}\mathrm{eil}}(A, W) \end{array}$$

is a homotopy pullback for any Weil-algebra A .

With x, y, a, b as in Lemma 1.8 we have, for each generator x_i of A , a homotopy pullback of the form

$$\begin{array}{ccc} B\Sigma_{n_{i,x}} \times B\Sigma_{n_{i,y}} & \longrightarrow & B\Sigma_{n_{i,a}} \times B\Sigma_{n_{i,b}} \times B\Sigma_{n_{i,ab}} \\ \downarrow & & \downarrow \\ * & \longrightarrow & B\Sigma_{n_{i,a}} \end{array}$$

where $n_{i,x} = n_{i,ab}$ and $n_{i,y} = n_{i,b}$, the top horizontal map matches up the corresponding terms, and the right-hand vertical map is the projection. Taking the product over all i and the disjoint union over all Weil-algebra morphisms $A \rightarrow W^2$, we get the desired homotopy pullback of mapping spaces. So the vertical lift diagram is a pullback in $\mathbb{W}\text{eil}$. \square

Similarly, we verify that the tensor product of Weil-algebras realizes the coproduct in the ∞ -category $\mathbb{W}\text{eil}$.

PROPOSITION 2.33. *Let A and A' be Weil-algebras. Then $A \otimes A'$ is the coproduct of A and A' in the ∞ -category $\mathbb{W}\text{eil}$.*

PROOF. The inclusion maps $A \rightarrow A \otimes A'$ and $A' \rightarrow A \otimes A'$ admit canonical labellings in the sense of Example 2.8. We check that for any Weil-algebra A'' , the induced map

$$\text{Hom}_{\mathbb{W}\text{eil}}(A \otimes A', A'') \rightarrow \text{Hom}_{\mathbb{W}\text{eil}}(A, A'') \times \text{Hom}_{\mathbb{W}\text{eil}}(A', A'')$$

is a homotopy equivalence. The tensor product is the coproduct in the homotopy category $\mathbb{W}\text{eil}_1$, so this map is a bijection on components. For a given Weil-algebra morphism $\phi : A \otimes A' \rightarrow A''$, the sequence of positive integers appearing in the expressions $\phi(x_i)$ for generators x_i of $A \otimes A'$ is clearly the disjoint union of the sequences for the corresponding maps $A \rightarrow A''$ and $A' \rightarrow A''$. The claim follows again by Lemma 2.29. \square

The Connection between $\mathbb{W}\text{eil}$ and \mathbb{E}_∞ . Our construction of the ∞ -category $\mathbb{W}\text{eil}$ was heavily motivated by Cranch's construction of what we are calling \mathbb{E}_∞ . However, there are even closer connections between the two ∞ -categories which we will describe in this section. First we fix our definition of \mathbb{E}_∞ .

DEFINITION 2.34. Let \mathbb{E}_∞ be the simplicial set in which an n -simplex is a J_n -indexed diagram of finite sets, i.e. a functor $\beta : J_n \rightarrow \text{FinSet}$, such that

- (1) for each $i < i' \leq j' < j$ in $[n]$, the square diagram

$$\begin{array}{ccc} & \beta([i, j]) & \\ \swarrow & & \searrow \\ \beta([i, j']) & & \beta([i', j]) \\ \searrow & & \swarrow \\ & \beta([i', j']) & \end{array}$$

is a pullback in FinSet .

We make \mathbb{E}_∞ into a strict monoidal ∞ -category with product \times and unit object the finite set $\{1\}$, following the same process as in the proof of Proposition 2.23 which appears in the section below.

DEFINITION 2.35. Let $U : \text{FPCM} \rightarrow \text{FinSet}$ be the forgetful functor, which sends a finite partial commutative monoid B to its underlying finite set. Then U preserves pullbacks, and so for any n -simplex $\alpha : J_n \rightarrow \text{FPCM}$ in the ∞ -category Weil , the functor $U\alpha : J_n \rightarrow \text{FinSet}$ is an n -simplex in \mathbb{E}_∞ . This construction gives us a functor $\mathcal{U} : \text{Weil} \rightarrow \mathbb{E}_\infty$, which can be viewed as sending the Weil-algebra A to its underlying additive commutative monoid (with basis given by the set M_A of nonzero monomials).

PROPOSITION 2.36. *Let $W : \text{FinSet} \rightarrow \text{FPCM}$ be the functor which sends a finite set J to the partial commutative monoid $M_{W^J} = J \sqcup \{1\}$. Composition with W determines a fully faithful functor $\mathcal{W} : \mathbb{E}_\infty \rightarrow \text{Weil}$ whose image is the subcategory of Weil generated by the Weil-algebras of the form W^J . Moreover \mathcal{W} preserves finite products.*

PROOF. First we show that composition with W takes an n -simplex in \mathbb{E}_∞ to an n -simplex in Weil , by verifying conditions (0)-(2) of Definition 2.17. For (0), we have already noted that $W(J) = M_{W^J}$. For (1), let $J \rightarrow K \leftarrow J'$ be a span of finite sets. Then $W(J) \rightarrow W(K) \leftarrow W(J')$ is a labelled Weil-algebra morphism from W^J to $W^{J'}$: the conditions of Definition 2.4 are easy to check in this case. Finally, for (2), we note that the addition of disjoint element preserves pullbacks of finite sets, so W preserves pullbacks. Thus there is a functor $\mathcal{W} : \mathbb{E}_\infty \rightarrow \text{Weil}$ as claimed.

To show that \mathcal{W} is fully faithful, it is sufficient to show that every n -simplex in Weil , whose vertices are of the form W^J , is equal to $W(\alpha)$ for an n -simplex α in \mathbb{E}_∞ . It is clear that the desired α must be given by deleting the identity element 1 from each finite partial commutative monoid. By Remark 2.9 that operation produces a diagram of finite sets and (fully-defined) functions between them. It is now sufficient to show that in any labelled morphism from W^J to $W^{J'}$ with vertex K , there cannot be any products defined in K which do not involve the identity element. But suppose $k \cdot k'$ is such a product. Then, again by Remark 2.9, $s(k)s(k')$ is a product in M_{W^J} which does not involve the identity element, a contradiction.

Finally note that finite products in \mathbb{E}_∞ are given by the disjoint union of finite sets; see [Bar17, Sec. 4]. It then follows from Proposition 2.31 that \mathcal{W} preserves those finite products. \square

A strict monoidal product on finite partial commutative monoids.

The purpose of this section is to prove Proposition 2.23 by giving an explicit construction of a strictly associative product \times on the category FPCM of finite partial commutative monoids. Our definition is based on the construction of Schauenberg in [Sch01, 4.3], which, in our case, depends on the construction below.

Throughout this section we write \boxtimes for the ordinary cartesian product of sets, and for the product in FPCM which is based on that cartesian product. That is, the elements of $B \boxtimes C$ are the ordered pairs (b, c) with $b \in B$ and $c \in C$. The partial monoid structure on $B \boxtimes C$ is as described in Remark 2.12.

DEFINITION 2.37. Given finite partial commutative monoids K_1, \dots, K_r , with $r \geq 1$ we write

$$[K_1, \dots, K_r] := \{(K_1, \dots, K_r)\} \boxtimes (K_1 \boxtimes (\dots \boxtimes (K_{r-1} \boxtimes K_r) \dots))$$

for the finite partial commutative monoid given by the iterated cartesian product of K_1, \dots, K_r with the one-element monoid whose one element is the sequence (K_1, \dots, K_r) itself. Recall that \boxtimes is not strictly associative, so we have to specify parentheses in this expression.

Notice that $[K_1, \dots, K_r]$ is isomorphic to a categorical product of K_1, \dots, K_r because the one-element object is a unit, up to isomorphism, for the cartesian product. In particular, for a single finite partial commutative monoid K , we have

$$[K] := \{(K)\} \boxtimes K,$$

which is canonically isomorphic to K via the projection map. We also take the empty sequence to represent the desired unit object:

$$[] := \{1\}.$$

Let FPCM' be the full subcategory of FPCM whose objects are the finite partial commutative monoids of the form $[K_1, \dots, K_r]$ for some (possibly empty) finite sequence K_1, \dots, K_r . It follows from the construction above that the objects of FPCM' are in one-to-one correspondence with finite sequences of objects in FPCM ; the purpose of including the additional one-element term consisting of the sequence itself is to ensure that we have that correspondence.

The rationale for the construction above is that the same object in FPCM can be given different decompositions, which need to be regarded as different for the purpose of defining the product \times below. For example, if $K = K_1 \boxtimes K_2$, then the objects $[K]$ and $[K_1, K_2]$, as defined above, are distinct, even though their ‘underlying’ objects in FPCM are the same.

DEFINITION 2.38. We define a product operation \times' on the subcategory FPCM' via concatenation of sequences:

$$[K_1, \dots, K_r] \times' [L_1, \dots, L_s] := [K_1, \dots, K_r, L_1, \dots, L_s].$$

This operation is well-defined because each object in FPCM' corresponds to a unique sequence. On morphisms \times' is determined by the universal property of the cartesian product \boxtimes used in the definition of the object $[K_1, \dots, K_r]$.

It is clear that \times' is a strict monoidal product on FPCM' , with strict unit $\{1\}$, that it models the categorical product, and that it also satisfies the desired unique decomposition property with

$$[K_1, \dots, K_r] = [K_1] \times' \dots \times' [K_r].$$

The indecomposable objects in FPCM' are precisely those of the form $[K]$ for some finite partial commutative monoid K .

We finish the proof of Proposition 2.23 by describing an isomorphism (not merely an equivalence) between FPCM and the full subcategory FPCM' , which we can use to transfer the operation \times' on to FPCM .

DEFINITION 2.39. Define a functor $\phi : \text{FPCM} \rightarrow \text{FPCM}'$ on objects by

$$\phi(K) := \begin{cases} [K] & \text{if } K \notin \text{FPCM}' \\ [K] & \text{if } K = [\dots [K'] \dots] \text{ for some } K' \notin \text{FPCM}' \\ K & \text{otherwise.} \end{cases}$$

The second case covers when K is a finite iteration of the construction of Definition 2.37 applied to a single object of FPCM which is not in FPCM' . For example,

if $K = [K']$ with $K' \notin \text{FPCM}'$, then $\phi(K) = [[K']]$. In general, ϕ increases by one the number of iterations for these objects. Note, however, that if $K = [K'_1, K'_2]$, then $\phi(K) = K$.

This ϕ is well-defined because any object of FPCM is either of the form $[K']$ or not, and if it is, then the object K' is uniquely determined. This definition provides a one-to-one correspondence between the objects of FPCM and the objects of FPCM'.

On morphisms ϕ is given either by the identity operation, or by composition with the canonical isomorphisms of the form $[K] \cong K$ mentioned above, and/or their inverses, depending on which of the cases above the source and target of a morphism fall under. These operations on morphisms are invertible, so ϕ is fully faithful. Therefore ϕ is an isomorphism of categories.

DEFINITION 2.40. We can now define our product operation \times on FPCM by

$$K \times L := \phi^{-1}(\phi(K) \times' \phi(L)).$$

PROOF OF 2.23. Since \times' is strictly associative, it follows that \times is too. Notice that $\phi(\{1\}) = \{1\}$, so $\{1\}$ is also a strict unit for \times . Finally, the unique decomposition property for \times' implies the corresponding property for \times . \square

CHAPTER 3

Tangent ∞ -Categories

We now turn back to tangent structures, and with the monoidal ∞ -category $\mathbb{W}eil$ in hand, we can extend Definition 1.9 to the ∞ -category case. To do so, we need an appropriate notion of monoidal functor for our monoidal ∞ -categories.

DEFINITION 3.1. A *strict monoidal functor* between two strict monoidal ∞ -categories \mathcal{A}^{\otimes} and \mathcal{B}^{\otimes} is a map of simplicial monoids $\mathcal{A} \rightarrow \mathcal{B}$, i.e. a map of simplicial sets that commutes with the monoid structures.

DEFINITION 3.2. Let \mathbb{X} be an ∞ -category. A *tangent structure* on \mathbb{X} is a strict monoidal functor

$$T : \mathbb{W}eil^{\otimes} \rightarrow \text{End}(\mathbb{X})^{\circ}$$

for which the underlying functor $T : \mathbb{W}eil \rightarrow \text{End}(\mathbb{X})$ preserves the tangent pullbacks of Proposition 2.31.

We will refer to the pair (\mathbb{X}, T) as a *tangent ∞ -category*. For brevity we often leave T understood and refer to ‘the tangent ∞ -category \mathbb{X} ’.

As with tangent categories, an individual Weil-algebra A determines an endofunctor $T^A : \mathbb{X} \rightarrow \mathbb{X}$, and we typically write $T = T^W$ for the *tangent bundle functor*, i.e. the endofunctor associated to the Weil-algebra $W = \mathbb{N}[x]/(x^2)$.

REMARK 3.3. We might expect the definition of tangent ∞ -category to be based on the more flexible notion of ‘strong’ monoidal functors. However, it turns out that any strong monoidal functor from $\mathbb{W}eil$ to another strict monoidal ∞ -category is equivalent to a strict monoidal functor. We prove that claim at the end of this chapter by establishing a cofibrancy property of the strict monoidal ∞ -category $\mathbb{W}eil^{\otimes}$; see Proposition 3.19.

REMARK 3.4. A tangent structure on an ∞ -category \mathbb{X} can equivalently be described via an *action map*, i.e. a map of simplicial sets

$$T : \mathbb{W}eil \times \mathbb{X} \rightarrow \mathbb{X}$$

which forms an action of the simplicial monoid $\mathbb{W}eil$ on the simplicial set \mathbb{X} . We will also say that \mathbb{X} is a *Weil-module*.

REMARK 3.5. Pullbacks in a functor ∞ -category such as $\text{End}(\mathbb{X})$ are calculated pointwise [Lur09a, 5.1.2.3], so the pullback condition in 3.2 can be expressed by saying that for each tangent pullback diagram in $\mathbb{W}eil$, and each object $C \in \mathbb{X}$, there is a certain pullback square in the ∞ -category \mathbb{X} . In particular, a tangent

structure on \mathbb{X} determines ‘vertical lift’ pullbacks in \mathbb{X} of the form

$$\begin{array}{ccc} TC \times_C TC & \longrightarrow & T^2C \\ \downarrow & & \downarrow \\ C & \longrightarrow & TC. \end{array}$$

REMARK 3.6. Much of Remark 1.12 extends to the ∞ -categorical case. The value of a tangent structure map $T : \text{Weil} \rightarrow \text{End}(\mathbb{X})$ on any Weil-algebra A is determined, up to equivalence, by the tangent bundle functor $T^W : \mathbb{X} \rightarrow \mathbb{X}$ which we usually denote also by T . For a positive integer n we write $T_n := T^{W^n}$ and for a finite set J we similarly write $T_J := T^{W^J}$.

A tangent structure on an ∞ -category \mathbb{X} also entails the five natural transformations $p, 0, +, c, \ell$ described in 1.12. Each of those natural transformations is based on a morphism of Weil-algebras which admits a canonical labelling which is essentially unique, as in Example 2.8, and hence determines a morphism in Weil . The tangent structure associates to that morphism a corresponding natural transformation between endofunctors on \mathbb{X} .

However, in place of the strictly commutative diagrams in Cockett and Crutwell’s definition of tangent category [CC14, 2.1], a tangent ∞ -category includes higher-level coherence data that establishes the commutativity of those diagrams up to homotopy.

For example, recall that a tangent structure on a category \mathbb{X} determines, for each $C \in \mathbb{X}$, the structure of a commutative monoid on the projection map $p : T(C) \rightarrow C$ within the slice category $\mathbb{X}/_C$. (In the language of [CC14, 2.1], p is an additive bundle over C .) In a similar way, a tangent structure on an ∞ -category \mathbb{X} determines the structure of an E_∞ -monoid on each projection map, that is an operation which is associative and commutative only up to higher coherent equivalences.

LEMMA 3.7. *Let \mathbb{X} be a tangent ∞ -category, and take an object $M \in \mathbb{X}$. The tangent structure determines the structure of an E_∞ -monoid on the projection map $p : T(M) \rightarrow M$ in the slice ∞ -category $\mathbb{X}/_M$.*

PROOF. Define a functor $\mathbb{E}_\infty \rightarrow \mathbb{X}/_M$ by the composite

$$\mathbb{E}_\infty \xrightarrow{\mathcal{W}} \text{Weil} \simeq \text{Weil}/_{\mathbb{N}} \xrightarrow{T} \text{End}(\mathbb{X})/_{I} \xrightarrow{\text{ev}/_M} \mathbb{X}/_M$$

of the functor \mathcal{W} of Proposition 2.36, the identification of Weil with the slice ∞ -category over its terminal object \mathbb{N} , the tangent structure map T (over \mathbb{N}), and the evaluation map at the object M . Each of these functors preserves finite products. (For T , that is a consequence of the preservation of the foundational pullbacks of Lemma 1.7.) Therefore the composite is a finite-product preserving functor from \mathbb{E}_∞ to $\mathbb{X}/_M$, i.e. an E_∞ -monoid in $\mathbb{X}/_M$, with underlying object given by the image of the one-element set, which maps to the projection map $p_M : T(M) \rightarrow M$. \square

LEMMA 3.8. *Let \mathbb{X} be an ordinary category. Then a tangent structure on (the nerve of) \mathbb{X} , in the sense of Definition 3.2, is the same thing as a tangent structure on the category \mathbb{X} in the sense of Definition 1.9.*

PROOF. The strict monoidal ∞ -category $\text{End}(\mathbb{X})^\circ$ can be identified with (the nerve of) the ordinary strict monoidal category $\text{End}(\mathbb{X})$. A map of simplicial sets

$\text{Weil} \rightarrow \text{End}(\mathbb{X})$ into the nerve of a category corresponds, by the adjunction between the nerve construction and taking the homotopy category, to an ordinary strict monoidal functor

$$\text{Weil}_1 \cong h\text{Weil} \rightarrow \text{End}(\mathbb{X}).$$

Since ∞ -categorical limits in the nerve of a category correspond to ordinary limits in the category, it follows that the two notions of tangent structure coincide. \square

WARNING 3.9. Every ∞ -category \mathbb{X} has an associated homotopy category $h\mathbb{X}$, and a tangent structure T on \mathbb{X} determines a strict monoidal functor on the level of homotopy categories

$$hT : \text{Weil}_1 \rightarrow \text{End}(h\mathbb{X}).$$

However, hT is typically *not* a tangent structure on $h\mathbb{X}$ since pullbacks in an ∞ -category \mathbb{X} do not usually determine pullbacks in $h\mathbb{X}$.

Examples of tangent ∞ -categories. Recall that a tangent structure on an ordinary category \mathbb{X} determines a tangent structure on \mathbb{X} viewed as an ∞ -category. However, there are also tangent structures on ∞ -categories that do not arise from ordinary categories. We start with some simple examples and constructions.

EXAMPLE 3.10. Let \mathbb{X} be an arbitrary ∞ -category. Then there is a tangent structure $I : \text{Weil}^\otimes \rightarrow \text{End}(\mathbb{X})^\circ$ given by the constant map to the identity functor on \mathbb{X} , the *trivial tangent structure* on \mathbb{X} .

EXAMPLE 3.11. Let (\mathbb{X}, T) be a tangent ∞ -category, and let S be any simplicial set. We give the ∞ -category $\text{Fun}(S, \mathbb{X})$, of S -indexed diagrams in \mathbb{X} , a tangent structure by extending T to a map

$$\text{Fun}(S, T) : \text{Weil} \times \text{Fun}(S, \mathbb{X}) \rightarrow \text{Fun}(S, \mathbb{X}),$$

which is a tangent structure because pullbacks in the ∞ -category $\text{Fun}(S, \mathbb{X})$ are determined objectwise.

LEMMA 3.12. *Let $i : \mathbb{X} \xrightarrow{\sim} \mathbb{Y}$ be an equivalence of ∞ -categories, and let $T : \mathbb{X} \rightarrow \mathbb{X}$ be the tangent bundle functor for a tangent structure on \mathbb{X} . Then there is a tangent structure on \mathbb{Y} whose underlying tangent bundle functor $\mathbb{Y} \rightarrow \mathbb{Y}$ is equivalent to iTi^{-1} .*

PROOF. We show that i induces an equivalence in the homotopy category of monoidal ∞ -categories

$$\text{End}(\mathbb{X})^\circ \simeq \text{End}(\mathbb{Y})^\circ$$

whose underlying functor is $i(-)i^{-1}$. This claim follows from the methods described in [Lur17, 4.7.1] in which a universal property is established for a monoidal ∞ -category of endomorphisms such as $\text{End}(\mathbb{X})^\circ$. That universal property says that $\text{End}(\mathbb{X})^\circ$ is the terminal object in the ∞ -category $\text{Cat}_\infty[\mathbb{X}]$ in which an object is a pair (\mathbb{F}, η) consisting of an ∞ -category \mathbb{F} and a functor $\eta : \mathbb{F} \rightarrow \text{End}(\mathbb{X})$. Any such terminal object has a unique associative algebra structure in $\text{Cat}_\infty[\mathbb{X}]$ which induces the structure of an associative algebra in Cat_∞ , i.e. a monoidal ∞ -category. (To be precise, this claim is an application of [Lur17, 4.7.1.40] with $\mathcal{C} = \mathcal{M} = \text{Cat}_\infty$, the ∞ -category of ∞ -categories, and $M = \mathbb{X}$.)

Now note that we have an equivalence of ∞ -categories

$$\text{Cat}_\infty[\mathbb{X}] \xrightarrow{\sim} \text{Cat}_\infty[\mathbb{Y}]; \quad (\mathbb{F}, \eta) \mapsto (\mathbb{F}, i(-)i^{-1} \circ \eta)$$

which therefore preserves the terminal objects and their algebra structures. In particular, $(\text{End}(\mathbb{X}), i(-)i^{-1})$ is a terminal object in $\text{Cat}_\infty[\mathbb{Y}]$, so is equivalent, as an associative algebra, to $(\text{End}(\mathbb{Y}), \text{id})$, yielding the desired monoidal equivalence.

Composing the given tangent structure map $T : \text{Weil}^\otimes \rightarrow \text{End}(\mathbb{X})^\circ$ with the equivalence constructed above determines the necessary tangent structure on the ∞ -category \mathbb{Y} . \square

We conclude this section with a tangent ∞ -category that extends the standard tangent structure on Mfld , the category of smooth manifolds and smooth maps.

Spivak introduced in [Spi10] an ∞ -category $\mathbb{D}\text{Mfld}$ of *derived manifolds*, constructed to allow for pullbacks along non-transverse pairs of smooth maps. We use a version of $\mathbb{D}\text{Mfld}$ that is characterized by the following universal property described by Carchedi and Steffens [CS19]; see there for more details.

DEFINITION 3.13. Let $\mathbb{D}\text{Mfld}$ denote an idempotent-complete ∞ -category with finite limits that admits a functor $i : \text{Mfld} \rightarrow \mathbb{D}\text{Mfld}$ with the following universal property: for any other idempotent-complete ∞ -category with finite limits \mathbb{C} , the functor i induces an equivalence

$$i^* : \text{Fun}^{\text{lex}}(\mathbb{D}\text{Mfld}, \mathbb{C}) \xrightarrow{\sim} \text{Fun}^{\text{th}}(\text{Mfld}, \mathbb{C})$$

between the ∞ -categories of finite-limit-preserving functors (on the left-hand side) and functors that preserve the *transverse pullbacks* and terminal object of Mfld (on the right-hand side). In particular, i preserves those transverse pullbacks and terminal object.

An explicit model for $\mathbb{D}\text{Mfld}$ is given by the opposite of the ∞ -category of (homotopically finitely presented) simplicial C^∞ -rings [CS19, 5.4].

LEMMA 3.14. *There is a monoidal functor*

$$\tilde{\bullet} : \text{Fun}^{\text{th}}(\text{Mfld}, \text{Mfld}) \rightarrow \text{Fun}^{\text{lex}}(\mathbb{D}\text{Mfld}, \mathbb{D}\text{Mfld})$$

induced by i along with the inverse to the equivalence of Definition 3.13.

PROOF. A formal argument can be made using ideas from [Lur17, 4.7.1] in a similar manner to the proof of Lemma 3.12. Here we sketch an informal approach. Given $F, G : \text{Mfld} \rightarrow \text{Mfld}$ that preserve the transverse pullbacks and terminal object, the universal property on $\mathbb{D}\text{Mfld}$ implies that iF and iG factor uniquely (up to contractible choice) as

$$iF \simeq \tilde{F}i, \quad iG \simeq \tilde{G}i$$

for $\tilde{F}, \tilde{G} : \mathbb{D}\text{Mfld} \rightarrow \mathbb{D}\text{Mfld}$. We then have

$$iFG \simeq \tilde{F}iG \simeq \tilde{F}\tilde{G}i$$

which implies that there is a canonical equivalence $\widetilde{FG} \simeq \tilde{F}\tilde{G}$. \square

PROPOSITION 3.15. *The standard tangent structure on Mfld induces, by composition with $\tilde{\bullet}$, a tangent structure on the ∞ -category $\mathbb{D}\text{Mfld}$.*

PROOF. First we argue that the tangent structure map

$$T : \text{Weil}_1 \rightarrow \text{Fun}(\text{Mfld}, \text{Mfld})$$

factors via $\text{Fun}^{\text{th}}(\text{Mfld}, \text{Mfld})$. It is sufficient to note that the tangent bundle functor $T : \text{Mfld} \rightarrow \text{Mfld}$ preserves the transverse pullbacks and terminal object, and this claim follows directly from the definition of transversality.

Composing T with the monoidal functor $\tilde{\bullet}$ described in Lemma 3.14, and with the truncation functor from Weil to its homotopy category Weil_1 , we obtain a monoidal functor

$$\tilde{T} : \text{Weil} \rightarrow \text{Fun}^{\text{lex}}(\mathbb{D}\text{Mfd}, \mathbb{D}\text{Mfd}).$$

Since the target of \tilde{T} is a full subcategory of $\text{End}(\mathbb{D}\text{Mfd})$, it is now sufficient to show that \tilde{T} preserves the tangent pullbacks. Equivalently, we must show that the composite

$$\text{Weil} \xrightarrow{T} \text{Fun}^{\text{h}}(\text{Mfd}, \text{Mfd}) \xrightarrow{i_*} \text{Fun}^{\text{h}}(\text{Mfd}, \mathbb{D}\text{Mfd})$$

preserves those pullbacks. The first map does (as T is a tangent structure on Mfd), and the second map does too (as each tangent pullback is transverse by [CC18, Ex. 4.4(ii)], and i preserves transverse pullbacks). \square

Cofibrancy of Weil . We now turn to the cofibrancy property of Weil which justifies our use of strict monoidal functors in the definition of tangent ∞ -categories in Definition 3.2. To describe that property, we develop in more detail the model for monoidal ∞ -categories and their monoidal functors that is outlined by Lurie in [Lur17, 4.1.8.7]. Underlying this model is a model structure on the category of ‘marked’ simplicial sets which forms a particularly nice foundation for the theory of ∞ -categories.

DEFINITION 3.16. A *marked simplicial set* consist of a pair (S, E) where S is a simplicial set and $E \subseteq S_1$ is a subset of the set of edges in S that contains all degenerate edges. We refer to E as a *marking* of S , and an element of E as a *marked edge*. Let Set_{Δ}^+ be the category of marked simplicial sets, with morphisms given by the maps of simplicial sets that preserve marked edges.

The category Set_{Δ}^+ admits a simplicial model structure [Lur09a, 3.1.3.7], which we refer to as the *marked model structure*, in which the cofibrations are precisely the monomorphisms, and the fibrant objects are those marked simplicial sets (S, E) for which S is an ∞ -category and E is the set of equivalences in S . The marked model structure on Set_{Δ}^+ is Quillen equivalent to the Joyal model structure on Set_{Δ} by [Lur17, 3.1.5] and so provides an alternative model for the homotopy theory of ∞ -categories. The marked model structure on Set_{Δ}^+ is also a monoidal model structure with respect to the cartesian product, in the sense of [Hov99, 4.2.6].

We will treat an ∞ -category S as a marked simplicial set without comment. Whenever we do so, we mean that the set E of marked edges is the set of equivalences in S . This choice is called the *natural marking* on S .

DEFINITION 3.17. A *marked simplicial monoid* is a simplicial monoid together with a marking of its underlying simplicial set, such that the product of marked edges is marked. Let Mon_{Δ}^+ denote the category whose objects are the marked simplicial monoids and whose morphisms are the maps of simplicial monoids that preserve the marking.

PROPOSITION 3.18. *There is a model structure on Mon_{Δ}^+ in which a morphism of marked simplicial monoids is a weak equivalence (or fibration) if and only if its underlying morphism of marked simplicial sets is a weak equivalence (or, respectively, a fibration) in the marked model structure.*

PROOF. The existence of this model structure is an application of [SS00, 4.1(3)]. See also [Lur17, 4.1.8.3]. \square

It follows from [Lur17, 4.1.8.4] that the model structure of Proposition 3.18 models the homotopy theory of all monoidal ∞ -categories (not just the strict ones) and all monoidal functors between them (not just the strict ones). In other words, monoidal functors from Weil^{\otimes} to $\text{End}(\mathbb{X})^{\circ}$ can be thought of as morphisms in Mon_{Δ}^+ , i.e. strict monoidal functors, from a cofibrant replacement of Weil^{\otimes} to the already-fibrant object $\text{End}(\mathbb{X})^{\circ}$.

Therefore, the fact that strict monoidal functors out of Weil^{\otimes} suffice to describe tangent structures on an ∞ -category \mathbb{X} is a consequence of the following calculation.

PROPOSITION 3.19. *The marked simplicial monoid Weil^{\otimes} (with only the identity morphisms in Weil as marked edges) is cofibrant in the model structure of Proposition 3.18.*

PROOF. The category Mon_{Δ} of (non-marked) simplicial monoids admits a model structure, due to Quillen [Qui67, II.4], in which the weak equivalences and fibrations are detected in the Quillen model structure on the underlying simplicial sets. The cofibrant objects in Mon_{Δ} are those simplicial monoids which are free in each simplicial degree, and for which the degeneracy operators preserve generators. We start by showing that Weil^{\otimes} has these two properties.

An n -simplex in the simplicial set Weil is a diagram in the category FPCM. Recall from Proposition 2.23 that any object in FPCM has a unique ordered decomposition

$$B = B_1 \times \cdots \times B_n$$

of B as a finite product in FPCM, where each B_i is neither itself a product nor is equal to the unit object $\{1\}$. Thus the monoid of objects in the strict monoidal category FPCM is freely generated by those indecomposable objects. In particular, recall we have chosen each Weil-algebra A to be represented by the partial commutative monoid

$$M_A = M_{W^{J_1}} \times \cdots \times M_{W^{J_r}}$$

where $A = W^{J_1} \otimes \cdots \otimes W^{J_r}$. So the vertices of the simplicial set Weil are freely generated by the objects W^J for nonempty finite sets J .

The n -simplexes in Weil are diagrams in FPCM indexed by the category \mathbf{J}_n of Definition 2.15. We will show that any diagram in FPCM can be uniquely written as an objectwise cartesian product of indecomposable diagrams; the particular category \mathbf{J}_n is not relevant here.

The decomposition of such a diagram $\alpha : \mathbf{J} \rightarrow \text{FPCM}$ as an objectwise product $\alpha = \beta \times \gamma$ is uniquely determined by a decomposition, for each vertex $j \in \mathbf{J}$, of $\alpha(j)$ as $\beta(j) \times \gamma(j)$. That decomposition exists if and only if for each morphism $j \rightarrow j'$ in \mathbf{J} , the function $\alpha(f)$ is equal to the product map $\beta(f) \times \gamma(f)$ for some $\beta(f) : \beta(j) \rightarrow \beta(j')$ and $\gamma(f) : \gamma(j) \rightarrow \gamma(j')$.

Suppose, for sake of a contradiction, that there is a diagram α which admits decompositions

$$\alpha = \beta \times \gamma = \beta' \times \gamma'$$

where β and β' are both indecomposable, and $\beta \neq \beta'$. For each object $j \in \mathbf{J}$, the object $\alpha(j)$ has a unique finite decomposition in FPCM of which $\beta(j)$ and $\beta'(j)$ are initial segments. Let $\beta''(j)$ denote the shorter of those two initial segments. Then we can write

$$\beta(j) = \beta''(j) \times \delta(j), \quad \beta'(j) = \beta''(j) \times \delta'(j)$$

where one of $\delta(j), \delta'(j)$ is equal to $\{1\}$ (depending on whether $\beta(j)$ or $\beta'(j)$ is the shorter of the initial segments), and the other is equal to the ‘difference’ between the two. For a particular j , we could have $\beta(j) = \beta'(j)$, in which case $\delta(j) = \delta'(j) = \{1\}$, but that is not the case for all j . Without loss of generality, we can assume that $\beta \neq \beta'$, so that there is some j for which $\delta(j)$ is non-trivial.

Our goal now is to prove that there is a non-trivial decomposition $\beta = \beta'' \times \delta$ as diagrams $J \rightarrow \text{FPCM}$, and for that we have to show that for each morphism $f : j \rightarrow j'$ in J , we have

$$\beta(f) = \beta''(f) \times \delta(f)$$

for some morphisms $\beta''(f) : \beta''(j) \rightarrow \beta''(j')$ and $\delta(f) : \delta(j) \rightarrow \delta(j')$ in FPCM . To see this claim, we separate into four cases depending on which of $\beta(j), \beta'(j)$ is shorter, and which of $\beta(j'), \beta'(j')$ is shorter. We use the notation $<$ and \leq to indicate this comparison:

- (1) $\beta(j) \leq \beta'(j)$ and $\beta(j') \leq \beta'(j')$: then we set

$$\beta''(f) = \beta(f) \quad \text{and} \quad \delta(f) = 1_{\{1\}}$$

and we have $\beta(f) = \beta''(f) \times \delta(f)$;

- (2) $\beta(j) \leq \beta'(j)$ and $\beta(j') > \beta'(j')$: then $\alpha(f)$ is a map of the form

$$\beta''(j) \times \delta'(j) \times \gamma'(j) \rightarrow \beta''(j') \times \delta(j') \times \gamma(j');$$

since $\alpha = \beta \times \gamma$, the $\beta''(j')$ and $\delta(j')$ components only depend on the $\beta''(j)$ input, and since $\alpha = \beta' \times \gamma'$, the $\delta(j')$ and $\gamma(j')$ components only depend on the $\gamma'(j)$ input; it follows that the $\delta(j')$ component is trivial; therefore we have

$$\beta(f) = \beta''(f) \times \delta(f)$$

where $\beta''(f)$ is $\beta(f)$ followed by the projection to $\beta''(j')$, and $\delta(f) : \{1\} \rightarrow \delta(j')$ is the trivial map;

- (3) $\beta(j) > \beta'(j)$ and $\beta(j') \leq \beta'(j')$: then $\alpha(f)$ is a map

$$\beta''(j) \times \delta(j) \times \gamma(j) \rightarrow \beta''(j') \times \delta'(j') \times \gamma'(j');$$

since $\alpha = \beta' \times \gamma'$, the $\beta''(j')$ and $\delta'(j')$ components only depend on the $\beta''(j) = \beta'(j)$ input, so we have

$$\beta(f) = \beta''(f) \times \delta(f)$$

where $\delta(f) : \delta(j) \rightarrow \{1\}$ is the trivial map;

- (4) $\beta(j) > \beta'(j)$ and $\beta(j') > \beta'(j')$: then $\alpha(f)$ is a map

$$\beta''(j) \times \delta(j) \times \gamma(j) \rightarrow \beta''(j') \times \delta(j') \times \gamma(j')$$

since $\alpha = \beta \times \gamma$, the $\beta''(j')$ output only depends on the $\beta''(j)$ input, and the $\delta(j')$ output only depends on the $\delta(j)$ and $\gamma(j)$ inputs; since $\alpha = \beta' \times \gamma'$, the $\delta(j')$ output also only depends on the $\beta''(j)$ and $\delta(j)$ inputs, so we must have that

$$\beta(f) = \beta''(f) \times \delta(f).$$

That completes the proof that $\beta = \beta'' \times \delta$, which contradicts the assumption that β is indecomposable. Hence no such α exists, and so every n -simplex in $\mathbb{W}eil$ is a unique product of indecomposable n -simplexes, as required.

Next suppose that $\alpha : J_n \rightarrow \text{FPCM}$ is an indecomposable n -simplex, and consider the $(n + 1)$ -simplex $\delta^j(\alpha) : J_{n+1} \rightarrow \text{FPCM}$. Each object or morphism in the diagram α is also an object or morphism in the diagram $\delta^j(\alpha)$, and so any non-trivial decomposition of $\delta^j(\alpha)$ determines a non-trivial decomposition of α . Therefore $\delta^j(\alpha)$ must also be indecomposable.

This completes our check of the two conditions from [Qui67, II.4], and so the simplicial monoid $\mathbb{W}eil^\otimes$ is cofibrant in Mon_Δ .

Let $(-)^b : \text{Set}_\Delta \rightarrow \text{Set}_\Delta^+$ be the ‘minimal’ marking functor, i.e. the functor which takes a simplicial set X to the marked simplicial set X^b which has only degenerate edges marked. The forgetful functor $U : \text{Set}_\Delta^+ \rightarrow \text{Set}_\Delta$ is right adjoint to $(-)^b$, and both U and $(-)^b$ are monoidal functors with respect to the cartesian product. It follows that $(-)^b$ and U induce an adjoint pair of functors, which we denote with the same names, on the categories of monoids

$$\text{Mon}_\Delta \begin{array}{c} \xrightarrow{(-)^b} \\ \xleftarrow{U} \end{array} \text{Mon}_\Delta^+.$$

We claim that the left adjoint $(-)^b : \text{Mon}_\Delta \rightarrow \text{Mon}_\Delta^+$ preserves cofibrations, and it is sufficient to show that the right adjoint $U : \text{Mon}_\Delta^+ \rightarrow \text{Mon}_\Delta$ preserves acyclic fibrations. Since those acyclic fibrations are detected in both cases in the underlying categories of (marked or not) simplicial sets, it is sufficient to check that $U : \text{Set}_\Delta^+ \rightarrow \text{Set}_\Delta$ preserves acyclic fibrations (where Set_Δ has the Quillen model structure). This claim follows from the fact that $(-)^b : \text{Set}_\Delta \rightarrow \text{Set}_\Delta^+$ preserves cofibrations (which are monomorphisms in both categories).

Since $\mathbb{W}eil$ is cofibrant in Mon_Δ , we deduce that $\mathbb{W}eil^b$ is cofibrant in Mon_Δ^+ , as claimed. \square

It follows from Lemma 3.19 that any morphism in the homotopy category of monoidal ∞ -categories, from $\mathbb{W}eil^\otimes$ to $\text{End}(\mathbb{X})^\circ$, can be represented by a strict map of marked simplicial monoids of the form

$$\mathbb{W}eil^b \rightarrow \text{End}(\mathbb{X})^\circ$$

or equivalently by a map of (non-marked) simplicial monoids $\mathbb{W}eil^\otimes \rightarrow \text{End}(\mathbb{X})^\circ$, as in Definition 3.2.

Tangent Functors

We now turn to morphisms between tangent structures on ∞ -categories. In the context of ordinary tangent categories, these are described by Cockett and Cruttwell in [CC14, 2.7]. They define a ‘strong’ morphism between tangent categories to be a functor on the underlying categories that commutes, up to natural isomorphism, with the tangent structures. They also have a notion of ‘lax’ morphism which involves a natural transformation that is not necessarily invertible.

Garner has shown in [Gar18, Thm. 9] how to express the Cockett-Cruttwell definition in terms of Weil-algebras. He shows that a (strong) morphism of tangent categories can, equivalently, be given via a map of ‘Weil₁-actegories’, i.e. a functor which commutes with the action of Weil₁ up to coherent isomorphism. The goal of this section is to extend that idea to tangent ∞ -categories.

Recall that we can view tangent ∞ -categories \mathbb{X} and \mathbb{Y} as ‘modules’ over the simplicial monoid $\mathbb{W}\text{eil}$. The most naive notion of tangent functor between them is therefore simply a map $\mathbb{X} \rightarrow \mathbb{Y}$ which commutes (strictly) with the actions of $\mathbb{W}\text{eil}$. However, we don’t usually expect a tangent functor to commute with the tangent structure maps on the nose. To build in the coherent isomorphisms described by Garner, we employ a model structure on the category of $\mathbb{W}\text{eil}$ -modules. In that model category, a tangent ∞ -category is a fibrant object. We define a tangent functor from \mathbb{X} to \mathbb{Y} to be a $\mathbb{W}\text{eil}$ -module homomorphism $B\mathbb{X} \rightarrow \mathbb{Y}$, where $B\mathbb{X}$ denotes a certain cofibrant replacement for \mathbb{X} in that model structure.

The $\mathbb{W}\text{eil}$ -module $B\mathbb{X}$ is given by a standard simplicial bar construction, and its structure encodes the isomorphisms appearing in Cockett and Cruttwell’s definition. We start by introducing the relevant model structure, and then we give an explicit definition of the bar construction $B\mathbb{X}$. The main result of this section is Proposition 4.13 which shows that our definition of tangent functor exactly recovers Cockett and Cruttwell’s when applied to tangent categories.

DEFINITION 4.1. A *marked Weil-module* consists of a marked simplicial set (recall from 3.16) together with a (strict) action of the marked simplicial monoid $\mathbb{W}\text{eil}$ (with the minimal marking). Let $\text{Mod}_{\mathbb{W}\text{eil}}^+$ denote the category of marked $\mathbb{W}\text{eil}$ -modules with morphisms given by those maps of marked simplicial sets that commute (strictly) with the action maps.

PROPOSITION 4.2. *There is a model structure on $\text{Mod}_{\mathbb{W}\text{eil}}^+$ in which the weak equivalences and fibrations are those maps of marked $\mathbb{W}\text{eil}$ -modules for which the underlying map of marked simplicial sets is a weak equivalence (respectively, a fibration) in the marked model structure on Set_{Δ}^+ . This model structure is enriched in the monoidal model category Set_{Δ}^+ of marked simplicial sets.*

PROOF. The model structure is an application of [SS00, 4.1(1)] to the marked model structure on Set_Δ^+ , and the enrichment result is an example of [LM07, 3.9]. \square

We use the model structure of Proposition 4.2 as our foundation for the appropriate notion of functor between tangent ∞ -categories. A tangent ∞ -category \mathbb{Y} naturally determines a fibrant marked Weil-module (with the equivalences in \mathbb{Y} as the marked edges). The appropriate notion of (strong) tangent functor $\mathbb{X} \rightarrow \mathbb{Y}$ between tangent ∞ -categories is then given by a map of marked Weil-modules from a cofibrant replacement of \mathbb{X} to \mathbb{Y} . To construct that cofibrant replacement, we employ a standard bar resolution using the free-forgetful adjunction between marked Weil-modules and marked simplicial sets.

Let $\theta : \text{Mod}_{\text{Weil}}^+ \rightarrow \text{Set}_\Delta^+$ denote the forgetful functor, and let its left adjoint $L : \text{Set}_\Delta^+ \rightarrow \text{Mod}_{\text{Weil}}^+$ defined by $L(X) := \text{Weil} \times X$ be the corresponding free functor. Associated to the monad θL is a bar construction for marked Weil-modules defined by May in [May72, 9.6].

DEFINITION 4.3. Let \mathbb{M} be a marked Weil-module. The *simplicial bar construction* on \mathbb{M} is the simplicial object in the category of marked Weil-modules

$$B_\bullet \mathbb{M} = B_\bullet(L, \theta L, \theta \mathbb{M})$$

with terms

$$B_k \mathbb{M} := (L\theta)^{k+1} \mathbb{M} = \text{Weil}^{k+1} \times \mathbb{M}$$

and with face and degeneracy maps determined by the counit and unit maps for the adjunction (L, θ) , i.e. by the multiplication and unit maps for the monoid Weil , and the action of Weil on \mathbb{M} .

Let $B\mathbb{M}$ denote the ‘realization’ of the simplicial object $B_\bullet \mathbb{M}$ in the sense of [Hir03, 18.6.2]. This is the marked Weil-module given by the coend

$$B\mathbb{M} := \int_{\Delta} \Delta^\bullet \times B_\bullet \mathbb{M}$$

and we will refer to $B\mathbb{M}$ as the *bar resolution* of \mathbb{M} . We can identify \mathbb{M} with a marked simplicial subset of $B\mathbb{M}$ via the unit map for the monoid Weil , but note that the inclusion $\mathbb{M} \subseteq B\mathbb{M}$ does not respect the Weil -actions.

It follows from [May72, 9.6] that the construction in Definition 4.3 determines a functor $B : \text{Mod}_{\text{Weil}}^+ \rightarrow \text{Mod}_{\text{Weil}}^+$.

REMARK 4.4. According to [Hir03, 15.11.6], the underlying simplicial set of the realization $B\mathbb{M}$ is given by the diagonal of the bisimplicial set $B_\bullet \mathbb{M}$. Thus a vertex in $B\mathbb{M}$ is a pair (A, M) consisting of a Weil-algebra A and object $M \in \mathbb{M}$, and an edge in $B\mathbb{M}$ from (A, M) to (A', M') consists of a triple (ϕ_0, ϕ_1, γ) where

$$\phi_0 : A \rightarrow A', \quad \phi_1 : A_1 \rightarrow A'_1$$

are labelled Weil-algebra morphisms,

$$\gamma : M_2 \rightarrow M'$$

is a morphism in \mathbb{M} , and $A'_0 \otimes A'_1 = A'$ and $A_1 \cdot M_1 = M$ (writing \cdot for the action of Weil on \mathbb{M}). The edge (ϕ_0, ϕ_1, γ) is marked in the marked Weil-module $B\mathbb{M}$ if each of its components is marked; i.e. ϕ_0, ϕ_1 are identity morphisms in Weil , and γ is marked in \mathbb{M} .

The counit of the adjunction (L, θ) induces an augmentation for the simplicial object $B_\bullet \mathbb{M}$ over \mathbb{M} , and hence a map of simplicial marked Weil-modules

$$\epsilon_\bullet : B_\bullet \mathbb{M} \rightarrow \mathbb{M}_\bullet$$

where \mathbb{M}_\bullet denotes the constant simplicial object with value \mathbb{M} . Taking realizations we get a map of marked Weil-modules

$$\epsilon : B\mathbb{M} \rightarrow \mathbb{M}.$$

LEMMA 4.5. *The map $\epsilon : B\mathbb{M} \rightarrow \mathbb{M}$ is a functorial cofibrant replacement for the marked Weil-module \mathbb{M} in the model structure of Proposition 4.2.*

PROOF. We have to show that ϵ is a weak equivalence, and that $B\mathbb{M}$ is cofibrant in $\text{Mod}_{\text{Weil}}^+$. First, by [May72, 9.8], the map of simplicial marked Weil-modules

$$\theta_\bullet \epsilon_\bullet : \theta_\bullet B_\bullet \mathbb{M} = B_\bullet(\theta L, \theta L, \theta \mathbb{M}) \rightarrow (\theta \mathbb{M})_\bullet,$$

given by applying θ levelwise to the map ϵ_\bullet , is a simplicial homotopy equivalence in the category of simplicial marked simplicial sets. Taking realizations, we deduce that $\theta\epsilon : \theta B\mathbb{M} \rightarrow \theta \mathbb{M}$ is a simplicial homotopy equivalence in the simplicial model category Set_Δ^+ . By [Hir03, 9.5.16], it follows that $\theta\epsilon$ is a weak equivalence in Set_Δ^+ , and hence that ϵ is a weak equivalence in $\text{Mod}_{\text{Weil}}^+$.

To show that $B\mathbb{M}$ is cofibrant in $\text{Mod}_{\text{Weil}}^+$, it is sufficient, by [Hir03, 18.6.7], to show that $B_\bullet \mathbb{M}$ is a Reedy cofibrant simplicial object, i.e. that each of the latching maps

$$L_k B_\bullet \mathbb{M} = \text{colim}_{[k] \rightarrow [i]} B_i \mathbb{M} \rightarrow B_k \mathbb{M}$$

is a cofibration in $\text{Mod}_{\text{Weil}}^+$. The morphisms in this colimit diagram are induced by the unit map for the monoid Weil , and the latching map is given by applying the free functor L to the inclusion into $\text{Weil}^k \times \mathbb{M}$ of the union of the marked simplicial subsets $\text{Weil}^i \times \{\mathbb{N}\} \times \text{Weil}^{k-1-i} \times \mathbb{M}$ for $i = 0, \dots, k-1$. The functor L is left adjoint to the right Quillen functor θ so is left Quillen and preserves cofibrations. Thus the latching map above is a cofibration as required. \square

We can now introduce a precise notion of tangent functor between tangent ∞ -categories.

DEFINITION 4.6. Let \mathbb{X} and \mathbb{Y} be tangent ∞ -categories. A *tangent functor* $\mathcal{F} : \mathbb{X} \rightarrow \mathbb{Y}$ is a map of marked Weil-modules

$$\mathcal{F} : B\mathbb{X} \rightarrow \mathbb{Y},$$

that is, a map of simplicial sets that commutes with the Weil-actions and preserves marked edges. The *underlying functor* of a tangent functor \mathcal{F} is given by restricting \mathcal{F} to the simplicial subset $\mathbb{X} \subseteq B\mathbb{X}$. We use the same notation \mathcal{F} for this underlying functor.

REMARK 4.7. If we drop the requirement that \mathcal{F} preserve marked edges, then we obtain a notion of *lax tangent functor*. Sometimes we will use the phrase *strong tangent functor* to distinguish the case where \mathcal{F} does preserve markings.

EXAMPLE 4.8. For any tangent ∞ -category \mathbb{X} , the canonical map $\epsilon : B\mathbb{X} \rightarrow \mathbb{X}$ is a tangent functor from \mathbb{X} to itself, which we refer to as the *identity tangent functor* on \mathbb{X} . The underlying functor for ϵ is the identity functor on \mathbb{X} .

DEFINITION 4.9. We write

$$\mathrm{Fun}^{\mathrm{tan}}(\mathbb{X}, \mathbb{Y}) \subseteq \mathrm{Fun}(B\mathbb{X}, \mathbb{Y})$$

for the simplicial subsets in which the n -simplexes are those maps of marked simplicial sets

$$B\mathbb{X} \rightarrow \mathrm{Fun}(\Delta^n, \mathbb{Y})$$

that commute with the Weil-actions, i.e. the tangent functors from \mathbb{X} to $\mathrm{Fun}(\Delta^n, \mathbb{Y})$ equipped with the tangent structure of Example 3.11. We refer to $\mathrm{Fun}^{\mathrm{tan}}(\mathbb{X}, \mathbb{Y})$ as the ∞ -category of tangent functors from $\mathbb{X} \rightarrow \mathbb{Y}$. That terminology is justified by the following result.

PROPOSITION 4.10. *Let \mathbb{X} and \mathbb{Y} be tangent ∞ -categories. The simplicial set $\mathrm{Fun}^{\mathrm{tan}}(\mathbb{X}, \mathbb{Y})$ is an ∞ -category, which is a category if \mathbb{Y} is a category.*

PROOF. The simplicial set $\mathrm{Fun}^{\mathrm{tan}}(\mathbb{X}, \mathbb{Y})$ is isomorphic to the underlying simplicial set of the hom-object

$$\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{W}\mathrm{eil}}^+}(B\mathbb{X}, \mathbb{Y})$$

in the Set_{Δ}^+ -enriched model category $\mathrm{Mod}_{\mathbb{W}\mathrm{eil}}^+$. Since $B\mathbb{X}$ is cofibrant, and \mathbb{Y} is fibrant, in that model structure, this hom-object is a fibrant marked simplicial set, i.e. an ∞ -category with the equivalences marked. Therefore $\mathrm{Fun}^{\mathrm{tan}}(\mathbb{X}, \mathbb{Y})$ is an ∞ -category.

Now suppose that \mathbb{Y} is an ordinary category. To see that $\mathrm{Fun}^{\mathrm{tan}}(\mathbb{X}, \mathbb{Y})$ is also a category, we have to show that every inner horn has a unique filler. Such a filler is a lift in a diagram of marked Weil-modules of the following form:

$$\begin{array}{ccc} & & \mathrm{Fun}(\Delta^n, \mathbb{Y}) \\ & \nearrow \text{dotted} & \downarrow i^* \\ B\mathbb{X} & \longrightarrow & \mathrm{Fun}(\Lambda_k^n, \mathbb{Y}) \end{array}$$

where $i : \Lambda_k^n \rightarrow \Delta^n$ is the inclusion of an inner horn. But since \mathbb{Y} is a category, the map i^* is an isomorphism, so every such diagram has a unique lift. Hence $\mathrm{Fun}^{\mathrm{tan}}(\mathbb{X}, \mathbb{Y})$ is a category whenever \mathbb{Y} is. \square

The goal of the remainder of this chapter is to connect Definition 4.6 to the Cockett-Crutwell notion of tangent functor. To do that, it is convenient to reinterpret our definition slightly. First note that the marked Weil-module $B\mathbb{W}\mathrm{eil}$ (given by applying Definition 4.3 to the action of $\mathbb{W}\mathrm{eil}$ on itself) can be viewed as a marked Weil-bimodule with right action determined by the action of $\mathbb{W}\mathrm{eil}$ on the rightmost copy of itself in the simplicial bar construction. For two (left) Weil-modules \mathbb{X} and \mathbb{Y} , the functor ∞ -category $\mathrm{Fun}(\mathbb{X}, \mathbb{Y})$ is also a Weil-bimodule with right action determined by the module structure on \mathbb{X} and left action by the module structure on \mathbb{Y} .

LEMMA 4.11. *Let \mathbb{X} and \mathbb{Y} be tangent ∞ -categories. There is a one-to-one correspondence between maps of (marked) Weil-modules $\mathcal{F} : B\mathbb{X} \rightarrow \mathbb{Y}$ and maps of (marked) Weil-bimodules*

$$\underline{\mathcal{F}} : B\mathbb{W}\mathrm{eil} \rightarrow \mathrm{Fun}(\mathbb{X}, \mathbb{Y})$$

where the marked edges in the ∞ -category $\mathrm{Fun}(\mathbb{X}, \mathbb{Y})$ are the equivalences.

PROOF. This claim is based on an isomorphism of marked Weil-modules

$$B\mathbb{X} \cong B\mathbb{W}\text{eil} \times_{\mathbb{W}\text{eil}} \mathbb{X}$$

where the right-hand side denotes a coend over the the actions of $\mathbb{W}\text{eil}$ on $B\mathbb{W}\text{eil}$ (on the right) and \mathbb{X} (on the left). That isomorphism in turn comes from isomorphisms at each simplicial level

$$\mathbb{W}\text{eil}^{k+1} \times \mathbb{X} \cong \mathbb{W}\text{eil}^{k+2} \times_{\mathbb{W}\text{eil}} \mathbb{X}$$

by taking the diagonal of these bisimplicial sets. \square

REMARK 4.12. We can use the formulation suggested by Lemma 4.11 to describe more explicitly the structure of a tangent functor \mathcal{F} between tangent ∞ -categories (\mathbb{X}, T) and (\mathbb{Y}, U) . Applying $\underline{\mathcal{F}}$ to the object (\mathbb{N}, \mathbb{N}) of $B\mathbb{W}\text{eil}$ yields the underlying functor $\mathcal{F} : \mathbb{X} \rightarrow \mathbb{Y}$. Applying $\underline{\mathcal{F}}$ to another object (A, A') of $B\mathbb{W}\text{eil}$ we obtain another functor which, since $\underline{\mathcal{F}}$ is a $\mathbb{W}\text{eil}$ -bimodule map, is necessarily given by the composite

$$U^A \mathcal{F} T^{A'} : \mathbb{X} \rightarrow \mathbb{Y}.$$

The edges in the simplicial set $B\mathbb{W}\text{eil}$ determine natural maps between these functors. In particular, applying $\underline{\mathcal{F}}$ to the morphism $e_A : (\mathbb{N}, A) \rightarrow (A, \mathbb{N})$ given by the triple of (labelled) Weil-algebra identity maps $(1_{\mathbb{N}}, 1_A, 1_{\mathbb{N}})$ yields a natural transformation

$$\alpha^A : \mathcal{F} T^A \rightarrow U^A \mathcal{F}.$$

The edge e_A is marked in $B\mathbb{W}\text{eil}$, so if \mathcal{F} is a strong tangent functor, the natural transformation α^A is required to be an equivalence in $\text{Fun}(\mathbb{X}, \mathbb{Y})$. These equivalences witness the requirement that the underlying functor \mathcal{F} commute with the tangent structures on \mathbb{X} and \mathbb{Y} , and the required higher coherences are indexed by the structure of the simplicial set $B\mathbb{W}\text{eil}$.

The description given in Remark 4.12 highlights the connection between our notion of tangent functor and the Cockett and Cruttwell's morphisms of tangent structure. Recall that a *strong morphism*, in [CC14, 2.7], between tangent categories (\mathbb{X}, T) and (\mathbb{Y}, U) consists of a pair (\mathcal{F}, α) where $\mathcal{F} : \mathbb{X} \rightarrow \mathbb{Y}$ is a functor which preserves the tangent pullbacks, and $\alpha : \mathcal{F} T \cong U \mathcal{F}$ is a natural isomorphism, subject to various compatibility conditions described there. A *tangent transformation* [Gar18, Def. 5] between strong tangent functors (\mathcal{F}, α) and (\mathcal{F}', α') is a natural transformation $\psi : \mathcal{F} \rightarrow \mathcal{F}'$ such that $(U\psi)\alpha = \alpha'(\psi T)$. The strong tangent morphisms from (\mathbb{X}, T) to (\mathbb{Y}, U) , together with these tangent transformations, form a category which we can now relate to our previous definitions.

PROPOSITION 4.13. *Let (\mathbb{X}, T) and (\mathbb{Y}, U) be tangent categories, viewed as tangent ∞ -categories via the identification of Lemma 3.8. Then $\text{Fun}^{\text{tan}}(\mathbb{X}, \mathbb{Y})$ is equivalent to the category whose objects are the strong morphisms of tangent categories and whose morphisms are the tangent transformations.*

PROOF. Garner proves in [Gar18, Thm. 9] that there is an equivalence between the category of Cockett and Cruttwell's strong morphisms of tangent structure, with tangent transformations as morphisms, and the category $\text{Fun}_{\mathbb{W}\text{eil}_1}(\mathbb{X}, \mathbb{Y})$ whose objects are functors of $\mathbb{W}\text{eil}_1$ -actegories from \mathbb{X} to \mathbb{Y} and whose morphisms are appropriate natural transformations. An object in $\text{Fun}_{\mathbb{W}\text{eil}_1}(\mathbb{X}, \mathbb{Y})$ consists of a functor $\mathcal{F} : \mathbb{X} \rightarrow \mathbb{Y}$ and natural isomorphisms

$$\alpha^A : \mathcal{F} T^A \xrightarrow{\cong} U^A \mathcal{F},$$

for each Weil-algebra A , such that the following diagrams in $\text{Fun}(\mathbb{X}, \mathbb{Y})$ commute:
for a Weil-algebra morphism $\phi : A \rightarrow A'$

$$(4.14) \quad \begin{array}{ccc} \mathcal{F}T^A & \xrightarrow{\alpha^A} & U^A \mathcal{F} \\ \mathcal{F}T^\phi \downarrow & & \downarrow U^\phi F \\ \mathcal{F}T^{A'} & \xrightarrow{\alpha^{A'}} & U^{A'} \mathcal{F} \end{array}$$

where we will denote the diagonal composite by α^ϕ , and for a pair of Weil-algebras A, A' :

$$(4.15) \quad \begin{array}{ccccc} \mathcal{F}T^A T^{A'} & \xrightarrow{\alpha^A T^{A'}} & U^A \mathcal{F} T^{A'} & \xrightarrow{U^A \alpha^{A'}} & U^A U^{A'} \mathcal{F} \\ \parallel & & & & \parallel \\ \mathcal{F}T^{A \otimes A'} & \xrightarrow{\alpha^{A \otimes A'}} & & & U^{A \otimes A'} \mathcal{F} \end{array}$$

and a unit condition

$$(4.16) \quad \begin{array}{ccc} & \mathcal{F} & \\ & \parallel & \parallel \\ \mathcal{F}T^{\mathbb{N}} & \xrightarrow{\alpha^{\mathbb{N}}} & U^{\mathbb{N}} \mathcal{F} \end{array}$$

A morphism in $\text{Fun}_{\text{Weil}_1}(\mathbb{X}, \mathbb{Y})$ from $(\mathcal{F}_0, \alpha_0)$ to $(\mathcal{F}_1, \alpha_1)$ consists of a natural transformation $\theta : \mathcal{F}_0 \rightarrow \mathcal{F}_1$ such that for each Weil-algebra A , the following diagram commutes:

$$(4.17) \quad \begin{array}{ccc} \mathcal{F}_0 T^A & \xrightarrow{\alpha_0^A} & U^A \mathcal{F}_0 \\ \theta T^A \downarrow & & \downarrow U^A \theta \\ \mathcal{F}_1 T^A & \xrightarrow{\alpha_1^A} & U^A \mathcal{F}_1 \end{array}$$

We will define an equivalence

$$(4.18) \quad e : \text{Fun}^{\text{tan}}(\mathbb{X}, \mathbb{Y}) \rightarrow \text{Fun}_{\text{Weil}_1}(\mathbb{X}, \mathbb{Y})$$

by sending the tangent functor $\underline{\mathcal{F}} : B\text{Weil} \rightarrow \text{Fun}(\mathbb{X}, \mathbb{Y})$ to the pair $(\mathcal{F}, \alpha^\bullet)$ given by

$$\mathcal{F} := \underline{\mathcal{F}}(\mathbb{N}, \mathbb{N}), \quad \alpha^A := \underline{\mathcal{F}}(1_{\mathbb{N}}, 1_A, 1_{\mathbb{N}})$$

where α^A is an isomorphism because $(1_{\mathbb{N}}, 1_A, 1_{\mathbb{N}})$ is a marked edge in $B\text{Weil}$. We verify the commutativity of each of the diagrams (4.14, 4.15, 4.16) by applying $\underline{\mathcal{F}}$ to an appropriate diagram in the simplicial set $B\text{Weil}$.

For (4.14), first choose a labelling for the Weil-algebra morphism ϕ so that it represents a 1-simplex in Weil . Then we get the desired diagram in $\text{Fun}(\mathbb{X}, \mathbb{Y})$ by

applying $\underline{\mathcal{F}}$ to a diagram in $B\mathbb{W}eil$ of the form

$$\begin{array}{ccc} (\mathbb{N}, A) & \xrightarrow{(1_{\mathbb{N}}, 1_A, 1_{\mathbb{N}})} & (A, \mathbb{N}) \\ \downarrow (1_{\mathbb{N}}, 1_{\mathbb{N}}, \phi) & \searrow & \downarrow (\phi, 1_{\mathbb{N}}, 1_{\mathbb{N}}) \\ (\mathbb{N}, A') & \xrightarrow{(1_{\mathbb{N}}, 1_{A'}, 1_{\mathbb{N}})} & (A', \mathbb{N}) \end{array}$$

where the top-right 2-simplex is the quadruple:

$$(1_{\mathbb{N}}, s^0(\phi), 1_{\mathbb{N}}, 1_{\mathbb{N}})$$

where $1_{\mathbb{N}}$ denotes the degenerate 2-simplex in $\mathbb{W}eil$ on the vertex \mathbb{N} , and $s^0(\phi)$ is the degenerate 2-simplex given by applying the degeneracy map s^0 to the 1-simplex represented by ϕ . Similarly, the bottom-left 2-simplex is the quadruple:

$$(1_{\mathbb{N}}, 1_{\mathbb{N}}, s^1(\phi), 1_{\mathbb{N}}).$$

To see that (4.15) commutes, apply $\underline{\mathcal{F}}$ to the 2-simplex

$$\begin{array}{ccc} & (A, A') & \\ (1_{\mathbb{N}}, 1_A, 1_{A'}) \nearrow & & \searrow (1_A, 1_{A'}, 1_{\mathbb{N}}) \\ (\mathbb{N}, A \otimes A') & \xrightarrow{(1_{\mathbb{N}}, 1_{A \otimes A'}, 1_{\mathbb{N}})} & (A \otimes A', \mathbb{N}) \end{array}$$

given by the quadruple

$$(1_{\mathbb{N}}, 1_A, 1_{A'}, 1_{\mathbb{N}})$$

consisting of the four degenerate 2-simplexes in $\mathbb{W}eil$ on each of these Weil-algebras. Finally, (4.16) commutes because $\alpha^{\mathbb{N}} = \underline{\mathcal{F}}(1_{\mathbb{N}}, 1_{\mathbb{N}}, 1_{\mathbb{N}})$ is given by applying $\underline{\mathcal{F}}$ to the degenerate 1-simplex $s^0(\mathbb{N})$, so yields a degenerate 1-simplex, i.e. an identity natural transformation, in $\text{Fun}(\mathbb{X}, \mathbb{Y})$. Thus $(\mathcal{F}, \alpha^\bullet)$ is a functor of $\mathbb{W}eil_1$ -actegories as claimed.

On morphisms our desired equivalence (4.18) is given by sending a Weil-bimodule map $\underline{\theta} : B\mathbb{W}eil \rightarrow \text{Fun}(\mathbb{X}, \text{Fun}(\Delta^1, \mathbb{Y}))$ to the natural transformation

$$\theta := \underline{\theta}(\mathbb{N}, \mathbb{N}).$$

The map $\underline{\theta}$ sends the 1-simplex $(1_{\mathbb{N}}, 1_A, 1_{\mathbb{N}})$ in $B\mathbb{W}eil$ to a 1-simplex in the ∞ -category $\text{Fun}(\mathbb{X}, \text{Fun}(\Delta^1, \mathbb{Y}))$, i.e. a square in $\text{Fun}(\mathbb{X}, \mathbb{Y})$ which provides the required commutative diagram (4.17).

To show that (4.18) is an equivalence of categories, we start by showing it is fully faithful. So suppose we have Weil-bimodule maps

$$\underline{\mathcal{F}}_0, \underline{\mathcal{F}}_1 : B\mathbb{W}eil \rightarrow \text{Fun}(\mathbb{X}, \mathbb{Y})$$

and a natural transformation

$$\theta : \underline{\mathcal{F}}_0(\mathbb{N}, \mathbb{N}) \rightarrow \underline{\mathcal{F}}_1(\mathbb{N}, \mathbb{N})$$

such that (4.17) commutes. We define a Weil-bimodule map

$$\underline{\theta} : B\mathbb{W}eil \rightarrow \text{Fun}(\mathbb{X}, \text{Fun}(\Delta^1, \mathbb{Y}))$$

which extends $\underline{\mathcal{F}}_0$ and $\underline{\mathcal{F}}_1$ by setting

$$\underline{\theta}(A, A') := U^A \theta T^{A'}$$

and on an edge (ϕ_0, ϕ_1, ϕ_2) in $B\text{Weil}$ consisting of labelled Weil-algebra morphisms $\phi_i : A_i \rightarrow A'_i$ by taking $\underline{\theta}(\phi_0, \phi_1, \phi_2)$ to be the following square in $\text{Fun}(\mathbb{X}, \mathbb{Y})$

$$\begin{array}{ccc} U^{A_0} \mathcal{F}_0 T^{A_1 \otimes A_2} & \xrightarrow{U^{\phi_0} \alpha_0^{\phi_1} T^{\phi_2}} & U^{A'_0 \otimes A'_1} \mathcal{F}_0 T^{A'_2} \\ \downarrow U^{A_0} \theta T^{A_1 \otimes A_2} & & \downarrow U^{A'_0 \otimes A'_1} \theta T^{A'_2} \\ U^{A_0} \mathcal{F}_1 T^{A_1 \otimes A_2} & \xrightarrow{U^{\phi_0} \alpha_1^{\phi_1} T^{\phi_2}} & U^{A'_0 \otimes A'_1} \mathcal{F}_1 T^{A'_2} \end{array}$$

which is commutative by (4.17). The value of $\underline{\theta}$ on higher-dimension simplexes is uniquely determined by the above information, and the chosen maps $\underline{\mathcal{F}}_0, \underline{\mathcal{F}}_1$ because $\text{Fun}(\mathbb{X}, \mathbb{Y})$ is a category. It follows that e is fully faithful.

To show that e is surjective on objects, suppose (\mathcal{F}, α) is a map of Weil_1 -categories from \mathbb{X} to \mathbb{Y} . We define a tangent functor

$$\underline{\mathcal{F}} : B\text{Weil} \rightarrow \text{Fun}(\mathbb{X}, \mathbb{Y})$$

by setting

$$\underline{\mathcal{F}}(A, A') := U^A \mathcal{F} T^{A'}$$

and, on edges by

$$\underline{\mathcal{F}}(\phi_0, \phi_1, \phi_2) := U^{A_0} \mathcal{F} T^{A_1 \otimes A_2} \xrightarrow{U^{\phi_0} \alpha^{\phi_1} T^{\phi_2}} U^{A'_0 \otimes A'_1} \mathcal{F} T^{A'_2}.$$

Note that this definition on edges ignores the labellings on each of the 1-simplexes ϕ_0, ϕ_1, ϕ_2 and only depends on the underlying Weil-algebra morphism. The definition also respects the face and degeneracy maps between vertices and edges by, among other things, (4.16).

A map of simplicial sets *into* a category, such as $\text{Fun}(\mathbb{X}, \mathbb{Y})$, is uniquely determined by where it sends vertices and edges, and an extension exists if and only if the edges of each 2-simplex map to three compatible morphisms in the category.

So take a quadruple $(\sigma_0, \sigma_1, \sigma_2, \sigma_3)$ where each σ_i is a 2-simplex in Weil , each of which has an underlying sequence of Weil-algebra morphisms of the form

$$\sigma_i : A_i^0 \xrightarrow{\sigma_i^1} A_i^1 \xrightarrow{\sigma_i^2} A_i^2.$$

We have to check that the following diagram in $\text{Fun}(\mathbb{X}, \mathbb{Y})$ commutes:

$$\begin{array}{ccc} & U^{A_0^1 \otimes A_1^1} \mathcal{F} T^{A_2^1 \otimes A_3^1} & \\ \begin{array}{c} \nearrow U^{\sigma_0^1} \alpha^{\sigma_1^1} T^{\sigma_2^1 \otimes \sigma_3^1} \\ \searrow U^{\sigma_0^2 \otimes \sigma_1^2} \alpha^{\sigma_2^2} T^{\sigma_3^2} \end{array} & & \\ U^{A_0^0} \mathcal{F} T^{A_1^0 \otimes A_2^0 \otimes A_3^0} & \xrightarrow{U^{\sigma_0^2 \sigma_1^2} \alpha^{\sigma_1^2 \sigma_2^2 \otimes \sigma_2^2 \sigma_3^2} T^{\sigma_3^2 \sigma_3^1}} & U^{A_0^2} \mathcal{F} T^{A_1^2 \otimes A_2^2 \otimes A_3^2} \end{array}$$

but this claim follows from (4.15) and several cases of (4.14).

The map $\underline{\mathcal{F}}$ so constructed is a map of Weil-bimodules. The marked edges in $B\text{Weil}$ are those of the form $(1_{A_0}, 1_{A_1}, 1_{A_2})$, and these are sent by $\underline{\mathcal{F}}$ to equivalences in $\text{Fun}(\mathbb{X}, \mathbb{Y})$ because α^{A_1} is an isomorphism. Therefore $\underline{\mathcal{F}}$ is a map of marked Weil-bimodules and corresponds by Lemma 4.11 to the desired tangent functor $\mathbb{X} \rightarrow \mathbb{Y}$. \square

REMARK 4.19. Proposition 4.13 extends to lax tangent functors by simply dropping the requirement that markings are preserved, and the condition that each α^A is an isomorphism. In particular, a lax tangent functor between tangent categories, in the sense described in Remark 4.7, determines a (not necessarily strong) ‘morphism of tangent structure’ in the sense of Cockett and Cruttwell [CC14, 2.7].

Differential Objects in Cartesian Tangent ∞ -Categories

Having established the basic definitions of tangent ∞ -categories and tangent functors, we now start to generalize the broader theory developed by Cockett and Cruttwell to the ∞ -categorical setting. In this chapter, we consider the notion of ‘differential object’ introduced in [CC14, Def. 4.8] (and strengthened slightly in [CC18, 3.1]) to describe the connection between tangent categories and cartesian differential categories.

Roughly speaking, the differential objects in a tangent category \mathbb{X} are the tangent *spaces*, that is the fibres $T_x M$ of the tangent bundle projections over (generalized) points $x : * \rightarrow M$ of objects in \mathbb{X} . In the tangent category of smooth manifolds these differential objects are the Euclidean vector spaces \mathbb{R}^n ; in the tangent category of schemes they are the affine spaces $\mathbb{A}^n = \text{Spec } \mathbb{Z}[x_1, \dots, x_n]$.

Our first goal in this chapter is to generalize Cockett and Cruttwell’s definition of differential object to tangent ∞ -categories. We do that by identifying a tangent ∞ -category which ‘represents’ differential objects in a tangent ∞ -category \mathbb{X} in the sense that those differential objects correspond to product-preserving tangent functors from that ∞ -category to \mathbb{X} .

We then prove Proposition 5.24, our analogue of [CC14, 4.15], which provides a canonical structure of a differential object on each tangent space in a tangent ∞ -category. In Corollary 5.26 we note that conversely any object that admits a differential structure is equivalent to a tangent space.

Finally in this chapter we turn to the connection between differential objects and the cartesian differential categories of Blute, Cockett and Seely [BCS09]. In 5.29, we prove a generalization of [CC14, 4.11] by showing that from every (cartesian) tangent ∞ -category \mathbb{X} we can construct a cartesian differential category whose objects are the differential objects of \mathbb{X} , and whose morphisms come from those in the homotopy category of \mathbb{X} .

Cockett and Cruttwell describe differential objects in the context of tangent categories that are *cartesian* in the following sense.

DEFINITION 5.1. A tangent ∞ -category, or tangent category, (\mathbb{X}, T) is *cartesian* if \mathbb{X} admits finite products (including a terminal object which we denote $*$), and the tangent bundle functor $T : \mathbb{X} \rightarrow \mathbb{X}$ preserves those finite products.

We now recall from [CC18, 3.1] the definition of differential object.

DEFINITION 5.2. A *differential object* in a cartesian tangent category (\mathbb{X}, T) consists of

- an object D in \mathbb{X} (the *underlying object*),

- morphisms $\sigma : D \times D \rightarrow D$ and $\zeta : * \rightarrow D$ that provide D with the structure of a commutative monoid in \mathbb{X} , and
- a morphism $\hat{p} : T(D) \rightarrow D$,

such that

- the map $\langle p, \hat{p} \rangle : T(D) \rightarrow D \times D$ is an isomorphism, and
- the five diagrams listed in [CC18, 3.1] commute.

A *morphism of differential objects* is a morphism of underlying objects that commutes with the differential structure maps σ , ζ and \hat{p} . We denote by $\mathbb{D}\text{iff}(\mathbb{X})$ the category of differential objects in \mathbb{X} and their morphisms.

WARNING 5.3. Cockett and Cruttwell in [CC14, 4.11] use the notation $\mathbb{D}\text{iff}(\mathbb{X})$ for a category whose morphisms include *all* morphisms between underlying objects, not only those that commute with the structure maps. We will write $\widehat{\mathbb{D}\text{iff}}(\mathbb{X})$ when we need to refer to that larger category.

In order to translate Definition 5.2 into the ∞ -categorical context, we first observe that the commutative monoid structure on a differential object should be replaced, in the context of ∞ -categories, with an E_∞ -structure. Recall that we described in Definition 2.34 an ∞ -category \mathbb{E}_∞ which acts as a Lawvere theory for E_∞ -monoids, in the sense that E_∞ -monoids in an ∞ -category \mathbb{X} correspond to finite-product-preserving functors $\mathbb{E}_\infty \rightarrow \mathbb{X}$. We should therefore anticipate that functors of this type will form part of the structure on a differential object in an ∞ -category.

In fact, there is a simple way to include the additional information needed to make an E_∞ -monoid $\mathbb{E}_\infty \rightarrow \mathbb{X}$ into a differential object in \mathbb{X} . We introduce a tangent structure on the ∞ -category \mathbb{E}_∞ , and then define a differential object in \mathbb{X} to be a (finite-product-preserving) *tangent* functor from \mathbb{E}_∞ to \mathbb{X} .

DEFINITION 5.4. Let $\otimes : \text{Weil} \times \mathbb{E}_\infty \rightarrow \mathbb{E}_\infty$ be the functor given by

$$\alpha \otimes \beta := \mathcal{U}(\alpha) \times \beta$$

for an n -simplex α in Weil , and an n -simplex β in \mathbb{E}_∞ , where $\mathcal{U} : \text{Weil} \rightarrow \mathbb{E}_\infty$ is induced by the forgetful functor $U : \text{FPCM} \rightarrow \text{FinSet}$ (see Definition 2.35), and \times is the objectwise (strict monoidal) product on diagrams in FinSet (constructed by the same process as in the proof of Proposition 2.23).

REMARK 5.5. An object of Weil is the finite partial commutative monoid M_A associated to a Weil-algebra A , and an object of \mathbb{E}_∞ is a finite set J where we often think of J as representing the free monoid \mathbb{N}^J . The choice of symbol \otimes for the functor in Definition 5.4 is motivated by the fact that the tensor product $A \otimes \mathbb{N}^J$ has a basis given by $U(M_A) \times J$.

PROPOSITION 5.6. *The functor $\otimes : \text{Weil} \times \mathbb{E}_\infty \rightarrow \mathbb{E}_\infty$ is a (strict) action of the simplicial monoid Weil^\otimes on the simplicial set \mathbb{E}_∞ , and provides \mathbb{E}_∞ with a tangent structure which we denote T_\otimes .*

PROOF. To see that \otimes is a strict action of Weil^\otimes on \mathbb{E}_∞ : let α, α' be n -simplexes in Weil , and let β be an n -simplex in \mathbb{E}_∞ ; then we have

$$(\alpha \otimes \alpha') \otimes \beta = \mathcal{U}(\alpha \otimes \alpha') \times \beta = \mathcal{U}(\alpha) \times \mathcal{U}(\alpha') \times \beta = \alpha \otimes (\alpha' \otimes \beta)$$

and

$$\mathbb{N} \otimes \beta = \mathcal{U}(M_{\mathbb{N}}) \times \beta = \{1\} \times \beta = \beta.$$

It remains then to show that \otimes preserves the tangent pullbacks.

For the foundational pullbacks (1.7), we have to show that for any finite sets J, J' , the following diagram is a homotopy pullback of spaces

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{E}_\infty}(J', M_{W^{m+n}} \times J) & \longrightarrow & \mathrm{Hom}_{\mathbb{E}_\infty}(J', M_{W^m} \times J) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathbb{E}_\infty}(J', M_{W^n} \times J) & \longrightarrow & \mathrm{Hom}_{\mathbb{E}_\infty}(J', J) \end{array}$$

This claim follows from the same type of argument as in the proof of Proposition 2.31. For the vertical lift pullback (1.8), we similarly have to show that for any finite sets J, J' , the following is a homotopy pullback

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{E}_\infty}(J', M_{W^2} \times J) & \longrightarrow & \mathrm{Hom}_{\mathbb{E}_\infty}(J', M_{W \otimes W} \times J) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathbb{E}_\infty}(J', J) & \longrightarrow & \mathrm{Hom}_{\mathbb{E}_\infty}(J', M_W \times J) \end{array}$$

and again this follows in the same way. \square

We can now give our definition of differential object in a tangent ∞ -category.

DEFINITION 5.7. Let (\mathbb{X}, T) be a cartesian tangent ∞ -category. A *differential object* in \mathbb{X} is a (strong) tangent functor

$$\mathcal{D} : (\mathbb{E}_\infty, T_\otimes) \rightarrow (\mathbb{X}, T)$$

for which the underlying functor $\mathbb{E}_\infty \rightarrow \mathbb{X}$ preserves finite products. We also say that \mathcal{D} is a *differential structure* on the object $\mathcal{D}(*)$, where $*$ denotes a one-element set.

We define the ∞ -category of differential objects in \mathbb{X} :

$$\mathbb{D}\mathrm{iff}(\mathbb{X}) \subseteq \mathrm{Fun}^{\mathrm{tan}}(\mathbb{E}_\infty, \mathbb{X})$$

to be the full subcategory whose objects are the tangent functors which preserve finite products.

Differential objects in a tangent category. The first main result of this chapter is to show that Definition 5.7 reduces to Definition 5.2 in the case that \mathbb{X} is an ordinary tangent category.

PROPOSITION 5.8. *Let (\mathbb{X}, T) be a cartesian tangent category. Then the ∞ -category $\mathbb{D}\mathrm{iff}(\mathbb{X})$ is an ordinary category, which is equivalent to the category of differential objects in \mathbb{X} and their morphisms, as described in Definition 5.2.*

PROOF. In the first part of the proof of Proposition 4.13 we show that when \mathbb{X} is an ordinary tangent category the simplicial set $\mathrm{Fun}^{\mathrm{tan}}(\mathbb{E}_\infty, \mathbb{X})$ is also a category. Therefore the full subcategory $\mathbb{D}\mathrm{iff}(\mathbb{X})$ is too.

Also, when \mathbb{X} is an ordinary category, any functor $\mathbb{E}_\infty \rightarrow \mathbb{X}$ factors uniquely via the homotopy category $h\mathbb{E}_\infty$, which is isomorphic to the category \mathbb{N}^\bullet whose objects are the free finitely-generated commutative monoids, and whose morphisms are the (unlabelled) monoid homomorphisms, i.e. the Lawvere theory for ordinary commutative monoids. It follows that there is an equivalence

$$\mathrm{Fun}^{\mathrm{tan}}(\mathbb{E}_\infty, \mathbb{X}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{tan}}(\mathbb{N}^\bullet, \mathbb{X})$$

where the tangent structure on the category \mathbb{N}^\bullet is given by the ordinary tensor product of commutative monoids. (Despite Warning 3.9, in this case the homotopy category *does* inherit a tangent structure since the tangent pullbacks in \mathbb{E}_∞ do pass to pullbacks in $h\mathbb{E}_\infty$.)

By Proposition 4.13 the right-hand side of the equivalence above can be identified with the category of (strong) tangent functors $\mathbb{N}^\bullet \rightarrow \mathbb{X}$ in the sense of [CC14], and the tangent transformations.

Thus our goal reduces to a claim purely within the ordinary tangent category theory of Cockett and Cruttwell: that differential objects in a tangent category \mathbb{X} in the sense of Definition 5.2 correspond to strong tangent functors $\mathbb{N}^\bullet \rightarrow \mathbb{X}$. We prove that fact in the next proposition. \square

PROPOSITION 5.9. *Let \mathbb{N}^\bullet be the category whose objects are the free finitely-generated commutative monoids (with a specified basis), and whose morphisms are the (unlabelled) monoid homomorphisms, with tangent structure given by the tensor product*

$$\otimes : \text{Weil}_1 \times \mathbb{N}^\bullet \rightarrow \mathbb{N}^\bullet.$$

Let \mathbb{X} be an arbitrary tangent category. Then there is an equivalence between the category of differential objects in \mathbb{X} , with morphisms which commute with the differential structure, and the category of strong tangent functors $\mathbb{N}^\bullet \rightarrow \mathbb{X}$, which preserve finite products, and their tangent natural transformations.

PROOF. To build the required equivalence, notice that the object \mathbb{N} in the category \mathbb{N}^\bullet has a differential structure given by its ordinary monoid structure, and with $\hat{p} : T_\otimes(\mathbb{N}) = W \otimes \mathbb{N} = W \rightarrow \mathbb{N}$ given by the monoid (but not algebra) map $a + bx \mapsto b$.

Any product-preserving tangent functor preserves differential objects, and the components of a tangent transformation are morphisms that commute with the differential structure maps σ , ζ and \hat{p} . We therefore have a functor e from the category of product-preserving tangent functors to the category of differential objects.

To see that e is fully faithful, take two product-preserving tangent functors $\mathcal{D}, \mathcal{D}' : \mathbb{N}^\bullet \rightarrow \mathbb{X}$, i.e. strong tangent morphisms (in the sense of [CC14, 2.7]) (\mathcal{D}, α) and (\mathcal{D}', α') . A tangent transformation $\beta : (\mathcal{D}, \alpha) \rightarrow (\mathcal{D}', \alpha')$ is a natural transformation $\beta : \mathcal{D} \rightarrow \mathcal{D}'$ such that the following diagram commutes

$$(5.10) \quad \begin{array}{ccc} \mathcal{D}T_\otimes & \xrightarrow{\alpha} & T\mathcal{D} \\ \beta T_{\mathbb{N}^\bullet} \downarrow & & \downarrow T\beta \\ \mathcal{D}'T_\otimes & \xrightarrow{\alpha'} & T\mathcal{D}' \end{array}$$

where T denotes the tangent bundle functor on \mathbb{X} .

Since all the functors in (5.10) preserve finite products, that diagram commutes if and only if it does so on its \mathbb{N} -component, and the natural transformation β is uniquely determined by the component $\beta_{\mathbb{N}} : \mathcal{D}(\mathbb{N}) \rightarrow \mathcal{D}'(\mathbb{N})$, which must be a map of commutative monoids in \mathbb{X} .

Therefore, the natural transformation β is a tangent transformation (i.e. the diagram above commutes) if and only if $\beta_{\mathbb{N}}$ commutes with the differential structure map \hat{p} . Thus e determines a bijection between the tangent transformations $(\mathcal{D}, \alpha) \rightarrow (\mathcal{D}', \alpha')$ and morphisms of differential objects $\mathcal{D}(\mathbb{N}) \rightarrow \mathcal{D}'(\mathbb{N})$.

To see that e is essentially surjective, consider a differential object $(D, \sigma, \zeta, \hat{p})$ in \mathbb{X} . The commutative monoid (D, σ, ζ) determines (uniquely up to isomorphism) a product-preserving functor $\mathcal{D} : \mathbb{N}^\bullet \rightarrow \mathbb{X}$ with $\mathcal{D}(\mathbb{N}^J) \cong D^J$.

To make \mathcal{D} into a tangent functor, we have to define a natural isomorphism

$$\alpha : \mathcal{D}T_\otimes \rightarrow T\mathcal{D}$$

that commutes with the tangent structure maps. We define the component

$$\alpha_{\mathbb{N}} : \mathcal{D}T_\otimes(\mathbb{N}) = \mathcal{D}(W) \cong D^{\{1,x\}} \xrightarrow{\sim} T(D) = T\mathcal{D}(\mathbb{N})$$

of α to be the inverse of the isomorphism $\langle p, \hat{p} \rangle$ of Definition 5.2. Other components are determined by the naturality requirement and the fact that all functors involved preserve finite products. Thus we obtain the natural isomorphism α .

To see that (\mathcal{D}, α) is a tangent functor, we check the conditions of [CC14, 2.7]. We will write out the proof for the commutative diagram involving the vertical lift ℓ ; the other conditions are much easier to verify. We must show that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{D}T_\otimes & \xrightarrow[\cong]{\alpha} & T\mathcal{D} \\ \mathcal{D}\ell_\otimes \downarrow & & \downarrow \ell_{\mathcal{D}} \\ \mathcal{D}T_\otimes^2 & \xrightarrow[\cong]{\alpha T_\otimes} T\mathcal{D}T_\otimes \xrightarrow[\cong]{T\alpha} & T^2\mathcal{D} \end{array}$$

Since all functors in this diagram preserve finite products, it is sufficient to look at the diagram of components at the object $\mathbb{N} \in \mathbb{N}^\bullet$. This diagram in \mathbb{X} takes the form

$$\begin{array}{ccc} D^2 & \xleftarrow[\cong]{\langle p, \hat{p} \rangle} & T(D) \\ \langle \pi_1, \zeta!, \zeta!, \pi_2 \rangle \downarrow & & \downarrow \ell \\ D^4 & \xleftarrow[\cong]{\langle p\pi_1, \hat{p}\pi_1, p\pi_2, \hat{p}\pi_2 \rangle} T(D)^2 \xleftarrow[\cong]{\langle T(p), T(\hat{p}) \rangle} & T^2(D) \end{array}$$

where we have identified the terms D^2 , $T(D)^2$ and D^4 by choosing an order on the basis elements of W (we use $\{1, x\}$) and $W \otimes W$ (we use $\{1, x_1, x_2, x_1x_2\}$), and we are writing ! for the unique map to the terminal object. We check that this diagram commutes by looking at each of the four components in turn:

$$pT(p)\ell = p0p = p, \quad \hat{p}T(p)\ell = \hat{p}0p = \zeta!, \quad pT(\hat{p})\ell = \hat{p}p_T\ell = \zeta!, \quad \hat{p}T(\hat{p})\ell = \hat{p}.$$

These equations follow from the definition of differential object in [CC18, 3.1]. In particular, the last equation is precisely the ‘extra’ axiom that was added to the definition of differential object in [CC18].

The remaining four diagrams in [CC14, 2.7] are verified in a similar manner. To see that (\mathcal{D}, α) is a strong morphism of tangent categories, we also need to show that the functor $\mathcal{D} : \mathbb{N}^J \mapsto D^J$ preserves the tangent pullbacks in \mathbb{N}^\bullet . Each of those pullbacks can be explicitly described using basis elements, and its preservation follows from the form of \mathcal{D} .

Finally we check that the differential structure on $e(\mathcal{D}) = \mathcal{D}(\mathbb{N}) = D$ induced by that on \mathbb{N} agrees with the original structure $(D, \sigma, \zeta, \hat{p})$. For the commutative monoid structure this is a standard fact about models for the Lawvere theory \mathbb{N}^\bullet . For the projection map \hat{p} we have to check that the map

$$T(D) = T\mathcal{D}(\mathbb{N}) \xrightarrow{\alpha_{\mathbb{N}}^{-1}} \mathcal{D}T_{\otimes}(\mathbb{N}) = \mathcal{D}(W) \xrightarrow{\mathcal{D}(\hat{p}_{\mathbb{N}})} \mathcal{D}(\mathbb{N}) = D$$

agrees with \hat{p} , which it does. Thus e is essentially surjective and is an equivalence of categories as claimed. \square

Differential structure on tangent spaces. The next main goal of this chapter is to identify differential objects in a tangent ∞ -category with the ‘tangent spaces’. This part of the paper parallels [CC14, Sec. 4.4].

DEFINITION 5.11. Let (\mathbb{X}, T) be a cartesian tangent ∞ -category. For a pointed object in \mathbb{X} , that is, a morphism $x : * \rightarrow C$ in \mathbb{X} where $*$ denotes a terminal object, the *tangent space to C at x* is the pullback

$$\begin{array}{ccc} T_x C & \longrightarrow & T(C) \\ \downarrow & & \downarrow p_C \\ * & \xrightarrow{x} & C \end{array}$$

if that pullback exists and is preserved by each $T^A : \mathbb{X} \rightarrow \mathbb{X}$.

We now wish to construct a differential object in \mathbb{X} whose underlying object is the tangent space $T_x C$, i.e. a product-preserving tangent functor $\mathcal{J}_x C : \mathbb{E}_{\infty} \rightarrow \mathbb{X}$ with $* \mapsto T_x C$. In fact, our construction will be functorial in x . That is, we will construct a functor

$$\mathcal{J}_{\bullet} : \mathbb{X}_{*}^T \rightarrow \mathbb{D}\text{iff}(\mathbb{X})$$

where \mathbb{X}_{*}^T is the ∞ -category of pointed objects in \mathbb{X} for which the tangent space exists.

The construction of \mathcal{J}_{\bullet} is complicated, so we give an outline first. Our strategy is to show first that there is a *lax* tangent functor $\mathcal{J}_{\mathbb{X}} C$, from \mathbb{E}_{∞} to \mathbb{X} , given on a finite set J by

$$\mathcal{J}_{\mathbb{X}} C(J) = T^{W^J}(C).$$

We then define $\mathcal{J}_x C : \mathbb{E}_{\infty} \rightarrow \mathbb{X}$ by taking the pullbacks of the projections $T^{W^J}(C) \rightarrow C$ over the point $x : * \rightarrow C$. The lax tangent structure on $\mathcal{J}_{\mathbb{X}} C$ then determines a tangent structure on the functor $\mathcal{J}_x C$ which turns out to be the desired differential structure on the tangent space.

The construction of $\mathcal{J}_{\mathbb{X}} C$ is founded on the case where $\mathbb{X} = \text{Weil}$ and $C = \mathbb{N}$, that is, on the universal example of an object in a tangent ∞ -category. In that case the underlying functor $\mathcal{J}\mathbb{N} : \mathbb{E}_{\infty} \rightarrow \text{Weil}$ is given by $J \mapsto W^J$, i.e. by the fully faithful inclusion $\mathcal{W} : \mathbb{E}_{\infty} \rightarrow \text{Weil}$ of Proposition 2.36. Our first task therefore is to define a lax tangent structure on \mathcal{W} , that is, to extend \mathcal{W} to a Weil-module map $\mathcal{W} : B\mathbb{E}_{\infty} \rightarrow \text{Weil}$.

As described in Remark 4.12, part of such a structure is a collection of natural transformations

$$\alpha_A : \mathcal{W}T_{\mathbb{E}_{\infty}}^A \rightarrow T_{\text{Weil}}^A \mathcal{W}$$

of functors $\mathbb{E}_\infty \rightarrow \mathbb{W}\text{eil}$. The component of α_A at an object J of \mathbb{E}_∞ (that is, at a finite set J) is a labelled Weil-algebra morphism

$$\alpha_J : \mathcal{W}(A \otimes J) \rightarrow A \otimes \mathcal{W}(J),$$

where \otimes in the source object refers to the action of $\mathbb{W}\text{eil}$ on \mathbb{E}_∞ described in Definition 5.4, and \otimes on the right-hand side is the tensor product of Weil-algebras. That map goes from the Weil-algebra $W^{M_A \times J}$ to the Weil-algebra $A \otimes W^J$ and is given by the following span in FPCM:

$$\begin{array}{ccc} & M_{W^{M_A \times J}} & \\ & \parallel & \searrow \iota_{M_A, J} \\ M_{W^{M_A \times J}} & & M_A \times W^J \end{array}$$

where the right-hand map is the map of partial commutative monoids

$$\iota_{M_A, J} : (M_A \times J) \sqcup \{1\} \rightarrow M_A \times (J \sqcup \{1\})$$

given by $(x, j) \mapsto (x, j)$, for $x \in M_A$ and $j \in J$, and $1 \mapsto (1, 1)$.

Similar maps to $\iota_{M_A, J}$ play a role in the full definition of the desired Weil-module map $B\mathbb{E}_\infty \rightarrow \mathbb{W}\text{eil}$.

DEFINITION 5.12. For $K \in \text{FPCM}$ and $J \in \text{FinSet}$, we have a map of partial commutative monoids

$$\iota_{K, J} : (K \times J) \sqcup \{1\} \rightarrow K \times (J \sqcup \{1\})$$

given by $(k, j) \mapsto (k, j)$, for $k \in K$ and $j \in J$, and $1 \mapsto (1, 1)$. Applying such maps objectwise to J_n -indexed diagrams in FPCM and FinSet , we get, for an n -simplex α in $\mathbb{W}\text{eil}$ and an n -simplex β in \mathbb{E}_∞ , a map

$$\iota(\alpha, \beta) : \mathcal{W}(\alpha \otimes \beta) \rightarrow \alpha \otimes \mathcal{W}(\beta)$$

between these two n -simplexes in $\mathbb{W}\text{eil}$, i.e. a diagram $\Delta^1 \times J_n \rightarrow \text{FPCM}$.

We now wish to extend the functor $\mathcal{W} : \mathbb{E}_\infty \rightarrow \mathbb{W}\text{eil}$ to a Weil-module map $\mathcal{W} : B\mathbb{E}_\infty \rightarrow \mathbb{W}\text{eil}$. Recall that we view \mathbb{E}_∞ as a simplicial subset of $B\mathbb{E}_\infty$ by identifying an n -simplex β in \mathbb{E}_∞ with the n -simplex in $B\mathbb{E}_\infty$ given by the $(n+2)$ -tuple

$$(1_{\mathbb{N}}, \dots, 1_{\mathbb{N}}, \beta)$$

where $1_{\mathbb{N}}$ denotes the degenerate n -simplex in $\mathbb{W}\text{eil}$ given by the constant diagram $J_n \rightarrow \text{FPCM}$ with value $\{1\}$.

DEFINITION 5.13. Take an n -simplex in $B\mathbb{E}_\infty$, i.e. an $(n+2)$ -tuple $(\alpha_0, \dots, \alpha_n, \beta)$, where $\alpha_0, \dots, \alpha_n$ are n -simplexes in $\mathbb{W}\text{eil}$, and β is an n -simplex in \mathbb{E}_∞ . We define

$$\mathcal{W}(\alpha_0, \dots, \alpha_n, \beta)$$

to be the n -simplex in $\mathbb{W}\text{eil}$, i.e. the diagram $J_n \rightarrow \text{FPCM}$, given on objects by

$$[i, j] \mapsto (\alpha_0 \otimes \dots \otimes \alpha_i \otimes \mathcal{W}(\alpha_{i+1} \otimes \dots \otimes \alpha_n \otimes \beta))([i, j])$$

and on morphisms, i.e. for intervals $[i', j'] \subseteq [i, j]$, by the composite

$$\begin{array}{ccc}
[i, j] & \longmapsto & (\alpha_0 \otimes \cdots \otimes \alpha_i \otimes \mathcal{W}(\alpha_{i+1} \otimes \cdots \otimes \alpha_n \otimes \beta))([i, j]) \\
\downarrow f & & \downarrow \iota(\alpha_{i+1} \otimes \cdots \otimes \alpha_{i'}, \alpha_{i'+1} \otimes \cdots \otimes \alpha_n \otimes \beta) \\
& & (\alpha_0 \otimes \cdots \otimes \alpha_{i'} \otimes \mathcal{W}(\alpha_{i'+1} \otimes \cdots \otimes \alpha_n \otimes \beta))([i, j]) \\
& & \downarrow (\alpha_0 \otimes \cdots \otimes \alpha_{i'} \otimes \mathcal{W}(\alpha_{i'+1} \otimes \cdots \otimes \alpha_n \otimes \beta))(f) \\
[i', j'] & \longmapsto & (\alpha_0 \otimes \cdots \otimes \alpha_{i'} \otimes \mathcal{W}(\alpha_{i'+1} \otimes \cdots \otimes \alpha_n \otimes \beta))([i', j'])
\end{array}$$

where $\iota(\alpha_{i+1} \otimes \cdots \otimes \alpha_{i'}, \alpha_{i'+1} \otimes \cdots \otimes \alpha_n \otimes \beta)$ is as in Definition 5.12.

LEMMA 5.14. *The construction of Definition 5.13 determines a Weil-module map $\mathcal{W} : B\mathbb{E}_\infty \rightarrow \mathbb{W}eil$, i.e. a lax tangent functor from \mathbb{E}_∞ to $\mathbb{W}eil$.*

PROOF. We first show that the definition above does indeed produce a functor

$$\mathcal{W}(\alpha_0, \dots, \alpha_n, \beta) : \mathbf{J}_n \rightarrow \mathbf{FPCM}.$$

Given $f : [i, j] \rightarrow [i', j']$ and $g : [i', j'] \rightarrow [i'', j'']$ there is a commutative diagram

$$\begin{array}{ccccc}
\cdots \mathcal{W}(\alpha_{i+1} \cdots)([i, j]) & & & & \\
\downarrow \iota & \searrow \iota & & & \\
\cdots \mathcal{W}(\alpha_{i'+1} \cdots)([i, j]) & \xrightarrow{\iota} & \cdots \mathcal{W}(\alpha_{i''+1} \cdots)([i, j]) & & \\
\downarrow \dots(f) & & \downarrow \dots(f) & \searrow \dots(gf) & \\
\cdots \mathcal{W}(\alpha_{i'+1} \cdots)([i', j']) & \xrightarrow{\iota} & \cdots \mathcal{W}(\alpha_{i''+1} \cdots)([i', j']) & \xrightarrow{\dots(g)} & \cdots \mathcal{W}(\alpha_{i''+1} \cdots)([i'', j''])
\end{array}$$

in which the top-left triangle and bottom-left square commute because of properties of the construction in Definition 5.12.

Next we check conditions (0)-(2) from Definition 2.17 to see that $\mathcal{W}(\alpha_0, \dots, \alpha_n, \beta)$ is an n -simplex in $\mathbb{W}eil$:

(0) We have

$$\mathcal{W}(\alpha_0, \dots, \alpha_n, \beta)([i, i]) = \alpha_0 \otimes \cdots \otimes \alpha_i \otimes \mathcal{W}(\alpha_{i+1} \otimes \cdots \otimes \alpha_n \otimes \beta)([i, i])$$

which is the set of nonzero monomials associated to a Weil-algebra of the form

$$A_0 \otimes \cdots \otimes A_i \otimes W^{M_{A_{i+1}} \times \cdots \times M_{A_n} \times B}$$

where $\alpha_k([i, i]) = M_{A_k}$, and $\beta([i, i]) = B$.

(1) It is sufficient to show that a span of the form

$$\begin{array}{ccc}
& \mathcal{W}(\alpha \otimes \beta)([i, j]) & \\
s \swarrow & & \searrow t \\
\mathcal{W}(\alpha \otimes \beta)([i, i]) & & \alpha \otimes \mathcal{W}(\beta)([j, j])
\end{array}$$

is a labelled Weil-algebra morphism for n -simplexes α, β in Weil and \mathbb{E}_∞ respectively. This span in turn is of the form

$$\begin{array}{ccc}
 & \mathbb{W}(\mathbb{U}(K) \times J'') & \\
 s \swarrow & & \searrow \iota \\
 \mathbb{W}(\mathbb{U}(M_A) \times J) & & K \times \mathbb{W}(J'') \\
 & & \searrow t' \times t'' \\
 & & M_{A'} \times \mathbb{W}(J')
 \end{array}$$

where $M_A \leftarrow K \xrightarrow{t'} M_{A'}$ is a labelled Weil-algebra morphism, and $J \leftarrow J'' \xrightarrow{t''} J'$ is a span of finite sets. We check the conditions of Definition 2.4:

(1) is satisfied because s is given by applying \mathbb{W} to a map of finite sets, see Proposition 2.36. For (2) suppose (k, j) and (k', j') are in $\mathbb{W}(\mathbb{U}(K) \times J'')$, and that their product in $M_A \times \mathbb{W}(J')$ is defined. Then $t''(j)t''(j')$ is defined in $\mathbb{W}(J')$, and so one of $t''(j)$ and $t''(j')$ must be the identity. So then either $\iota(k, j)$ or $\iota(k', j')$ is of the form $(-, 1)$ in $K \times \mathbb{W}(J'')$. But then by Definition 5.12, we must have either (k, j) or (k', j') equal to the identity element in $\mathbb{W}(\mathbb{U}(K) \times J')$, which means that the product of (k, j) and (k', j') is defined.

(2) It is sufficient to show that the big square in the following diagram is a pullback in FPCM, i.e. a pullback of finite sets:

$$\begin{array}{ccccc}
 & & \mathbb{W}(\alpha \otimes \beta)([i, l]) & & \\
 & \swarrow & & \searrow \iota(\alpha, \beta) & \\
 \mathbb{W}(\alpha \otimes \beta)([i, k]) & & & & \alpha \otimes \mathbb{W}(\beta)([i, l]) \\
 & \searrow \iota(\alpha, \beta) & & \swarrow & \searrow \\
 & & \alpha \otimes \mathbb{W}(\beta)([i, k]) & & \alpha \otimes \mathbb{W}(\beta)([j, l]) \\
 & & \swarrow & & \swarrow \\
 & & & & \alpha \otimes \mathbb{W}(\beta)([j, k])
 \end{array}$$

is a pullback of finite sets. The bottom-right square is a pullback because $\alpha \otimes \mathbb{W}(\beta)$ is an n -simplex in Weil . The top-left square is of the form

$$\begin{array}{ccc}
 (K \times J) \sqcup \{1\} & & \\
 \swarrow & & \searrow \\
 (K' \times J') \sqcup \{1\} & & K \times (J \sqcup \{1\}) \\
 \swarrow & & \searrow \\
 & & K' \times (J \sqcup \{1\})
 \end{array}$$

where $f : K \rightarrow K'$ is some partial monoid homomorphism, and $J \rightarrow J'$ is some map of finite sets. The downward-right maps are each the inclusion with 1 mapping to $(1, 1)$. To show this diagram is a pullback of finite sets, it is sufficient to see that $f^{-1}(1) = \{1\}$. By Remark 2.9, the composite

partial monoid homomorphism

$$\alpha([i, l]) \xrightarrow{f} \alpha([i, k]) \xrightarrow{g} \alpha([i, i])$$

has the property that only 1 maps to 1, and so the same is true of f .

We next verify that our chosen n -simplexes $\mathcal{W}(\alpha_0, \dots, \alpha_n, \beta)$ form a map of simplicial sets $\mathcal{W} : B\mathbb{E}_\infty \rightarrow \text{Weil}$. For an order-preserving function $q : [m] \rightarrow [n]$, we have

$$q^*(\alpha_0, \dots, \alpha_n, \beta) = (q^*\alpha_0 \otimes \dots \otimes q^*\alpha_{q(0)}, \dots, q^*\alpha_{q(m)+1} \otimes \dots \otimes q^*\beta)$$

and so

$$\mathcal{W}(q^*(\alpha_0, \dots, \alpha_n, \beta))([i, j]) = (\alpha_0 \otimes \dots \otimes \alpha_{q(i)} \otimes \mathcal{W}(\alpha_{q(i)+1} \otimes \dots \otimes \beta))([q(i), q(j)])$$

which is equal to $\mathcal{W}(\alpha_0, \dots, \alpha_n, \beta)([q(i), q(j)])$, and hence to $q^*\mathcal{W}(\alpha_0, \dots, \alpha_n, \beta)$, as required.

Finally, we note that \mathcal{W} is a map of Weil-modules since

$$\alpha \otimes \mathcal{W}(\alpha_0, \alpha_1, \dots, \alpha_n, \beta) = \mathcal{W}(\alpha \otimes \alpha_0, \alpha_1, \dots, \alpha_n, \beta).$$

This completes the proof that $\mathcal{W} : B\mathbb{E}_\infty \rightarrow \text{Weil}$ determines a lax tangent functor from \mathbb{E}_∞ to Weil . \square

DEFINITION 5.15. Let \mathbb{X} be a tangent ∞ -category. We define a functor

$$\mathcal{J}_\mathbb{X} : \mathbb{X} \rightarrow \text{Fun}_{\text{lax}}^{\text{tan}}(\mathbb{E}_\infty, \mathbb{X})$$

to be adjunct to the composite map (of Weil-modules):

$$B\mathbb{E}_\infty \times \mathbb{X} \xrightarrow{W \times 1} \text{Weil} \times \mathbb{X} \xrightarrow{T} \mathbb{X}.$$

For each object C of \mathbb{X} , we therefore have a lax tangent functor $\mathcal{J}_\mathbb{X}C : \mathbb{E}_\infty \rightarrow \mathbb{X}$, with underlying functor given by

$$(\mathcal{J}_\mathbb{X}C)(J) = T^{W^J}(C).$$

The next step is to form the pullback of the functor $\mathcal{J}_\mathbb{X}C$ over some ‘point’ $x : * \rightarrow C$ in the object C to recover the tangent *space* to C at x . For that we introduce the following ‘constant’ lax tangent functors.

DEFINITION 5.16. Let $\mathcal{Z} : \mathbb{E}_\infty \rightarrow \mathbb{E}_\infty$ denote the constant functor with value the empty set \emptyset . (Note that the object \emptyset is both initial and terminal in the ∞ -category \mathbb{E}_∞ , with the unique labelled morphism either to or from \emptyset given by the span whose set of labels is \emptyset too.) The functor \mathcal{Z} strictly commutes with the Weil-action on \mathbb{E}_∞ because $\mathbf{U}(K) \times \emptyset = \emptyset$ for any finite partial commutative monoid K . Therefore, \mathcal{Z} induces a Weil-module map

$$B\mathcal{Z} : B\mathbb{E}_\infty \rightarrow B\mathbb{E}_\infty.$$

Let $\mathcal{K}_\mathbb{X} : \mathbb{X} \rightarrow \text{Fun}_{\text{lax}}^{\text{tan}}(\mathbb{E}_\infty, \mathbb{X})$ be the functor given by

$$\mathcal{K}_\mathbb{X}C := \mathcal{J}_\mathbb{X}C \circ B\mathcal{Z}.$$

The underlying functor of $\mathcal{K}_\mathbb{X}C$ is the constant functor $\mathbb{E}_\infty \rightarrow \mathbb{X}$ with value C .

In order to form the pullback of TC along the map $x : * \rightarrow C$, in the ∞ -category of lax tangent functors $\mathbb{E}_\infty \rightarrow \mathbb{X}$, we also need a natural transformation $\mathcal{J}_\mathbb{X}C \rightarrow \mathcal{K}_\mathbb{X}C$ which overlies the projection $p_C : TC \rightarrow C$.

DEFINITION 5.17. We define a natural transformation $\epsilon : \Delta^1 \times \mathbb{E}_\infty \rightarrow \mathbb{E}_\infty$ from the identify functor to \mathcal{Z} as follows. Given an n -simplex $\delta_k = (\underbrace{0, \dots, 0}_k, \underbrace{1, \dots, 1}_{n+1-k})$ in Δ^1 , and an n -simplex $\beta : \mathbb{J}_n \rightarrow \text{FinSet}$ in \mathbb{E}_∞ , we define $\epsilon(\delta_k, \beta)$ to be the n -simplex in \mathbb{E}_∞ given on objects by

$$\epsilon(\delta_k, \beta)([i, j]) := \begin{cases} \beta([i, j]) & \text{if } k > j; \\ \emptyset & \text{if } k \leq j. \end{cases}$$

On morphisms, i.e. for $[i', j'] \subseteq [i, j]$, the relevant map is $\beta([i, j]) \rightarrow \beta([i', j'])$ if $k > j$ (and hence $k > j'$), and the unique map out of the empty set otherwise.

In particular, the component of ϵ at the finite set J is the morphism in \mathbb{E}_∞ given by the span

$$\begin{array}{ccc} & \emptyset & \\ & \swarrow & \searrow \\ J & & \emptyset \end{array}$$

which represents the unique map from J to the terminal object \emptyset .

The pullback condition for $\epsilon(\delta_k, \beta)$ follows from that for β together with the fact that any diagram of the form

$$\begin{array}{ccc} & \emptyset & \\ & \swarrow & \searrow \\ \beta([i, j']) & & \emptyset \\ & \searrow & \swarrow \\ & \beta([i', j']) & \end{array}$$

is a pullback of finite sets.

LEMMA 5.18. *Let \mathbb{X} be a tangent ∞ -category, and let C be an object in \mathbb{X} . Then the map $\epsilon : \Delta^1 \times \mathbb{E}_\infty \rightarrow \mathbb{E}_\infty$ determines a tangent natural transformation, between lax tangent functors $\mathbb{E}_\infty \rightarrow \mathbb{X}$:*

$$\epsilon^* : \mathcal{T}_{\mathbb{X}} C \rightarrow \mathcal{K}_{\mathbb{X}} C$$

whose underlying natural transformation has components $T^{W^J}(C) \rightarrow C$ induced by the unique maps of Weil-algebras $W^J \rightarrow \mathbb{N}$, for finite sets J .

PROOF. The map ϵ commutes with the Weil-action on \mathbb{E}_∞ because

$$\alpha \otimes \epsilon(\delta_k, \beta) = \epsilon(\delta_k, \alpha \otimes \beta)$$

for n -simplexes α in Weil , δ_k in Δ^1 , and β in \mathbb{E}_∞ . Therefore ϵ induces a map $B_\epsilon : \Delta^1 \times B\mathbb{E}_\infty \rightarrow B\mathbb{E}_\infty$. The desired map $\epsilon^* : \Delta^1 \rightarrow \text{Fun}_{\text{lx}}^{\text{tan}}(\mathbb{E}_\infty, \mathbb{X})$ is then adjoint to the composite

$$\Delta^1 \times B\mathbb{E}_\infty \xrightarrow{B_\epsilon} B\mathbb{E}_\infty \xrightarrow{\mathcal{T}_{\mathbb{X}} C} \mathbb{X}.$$

□

Now suppose \mathbb{X} is a cartesian tangent ∞ -category, and let $x : * \rightarrow C$ be a pointed object in \mathbb{X} for which the tangent space $T_x C$ exists in the sense of Definition 5.11. Then we have a diagram in $\text{Fun}_{\text{lax}}^{\text{tan}}(\mathbb{E}_\infty, \mathbb{X})$ given by

$$(5.19) \quad \begin{array}{ccc} & & \mathcal{J}_{\mathbb{X}} C \\ & & \downarrow \epsilon^* \\ \mathcal{K}_{\mathbb{X}}(*) & \xrightarrow{\mathcal{K}_{\mathbb{X}}(x)} & \mathcal{K}_{\mathbb{X}} C \end{array}$$

for which the underlying diagram of functors $\mathbb{E}_\infty \rightarrow \mathbb{X}$ takes the form

$$(5.20) \quad \begin{array}{ccc} & & T_J(C) \\ & & \downarrow p_C \\ * & \xrightarrow{x} & C \end{array}$$

In particular, when $J = *$ we recover the cospan diagram whose pullback is the tangent space $T_x C$. These constructions are all functorial in the pointed object x , so overall we can make the following definition.

DEFINITION 5.21. Diagrams of the form (5.19) determine a functor

$$\mathcal{J}_{\sqcup} : \mathbb{X}_* \rightarrow \text{Fun}_{\text{lax}}^{\text{tan}}(\mathbb{E}_\infty, \text{Fun}(\sqcup, \mathbb{X}))$$

from the ∞ -category of pointed objects in \mathbb{X} to the ∞ -category of (lax) tangent functor from \mathbb{E}_∞ to the ∞ -category of cospan diagrams $\text{Fun}(\sqcup, \mathbb{X})$ with the pointwise tangent structure induced by that on \mathbb{X} as in Example 3.11. Here $\sqcup = (0 \rightarrow 01 \leftarrow 1)$ is the indexing category for cospan diagrams.

We now wish to compose with a functorial pullback to get tangent functors that take values in \mathbb{X} . However, we are not assuming that \mathbb{X} has all pullbacks, so we first make the following definition.

DEFINITION 5.22. Let $\mathbb{Y} \subseteq \text{Fun}(\sqcup, \mathbb{X})$ be the full subcategory on those cospan diagrams in \mathbb{X} that admit a pullback *and* for which that pullback is preserved by T^A for all Weil-algebras A . Since \mathbb{Y} is invariant under the action of Weil , and pullbacks in \mathbb{Y} are calculated objectwise, it is clear that \mathbb{Y} is a tangent subcategory of $\text{Fun}(\sqcup, \mathbb{X})$.

LEMMA 5.23. *Let \mathbb{X} be a cartesian tangent ∞ -category. Then the functor \mathcal{J}_{\sqcup} of Definition 5.21 restricts to a functor*

$$\mathcal{J}_{\sqcup} : \mathbb{X}_*^T \rightarrow \text{Fun}_{\text{lax}}^{\text{tan}}(\mathbb{E}_\infty, \mathbb{Y})$$

where \mathbb{X}_*^T is the full subcategory of pointed objects in \mathbb{X} for which the corresponding tangent space $T_x C$ exists in the sense of Definition 5.11.

PROOF. We have to show that for any Weil-algebra A , and any finite set J , the diagram (5.19), when evaluated at (A, J) , is in \mathbb{Y} , i.e. admits a pullback which

is preserved by $T^{A'}$ for any Weil-algebra A' . Those diagrams are of the form

$$\begin{array}{ccc} & T^A T^{W^J}(C) & \\ & \downarrow T^A(p_C) & \\ T^A(*) & \xrightarrow{T^A(x)} & T^A(C) \end{array}$$

It is clearly sufficient to consider the case $A = \mathbb{N}$ when the diagram takes the form

$$\begin{array}{ccc} & T_k C & \\ & \downarrow p_C & \\ * & \xrightarrow{x} & C \end{array}$$

for some $k \geq 0$. When $k = 0$, the map labelled p_C is the identity on C , and the pullback is $*$. When $k = 1$, pullback is precisely the tangent space $T_x C$. In both of these cases, that pullback is preserved by $T^{A'}$ for all A' . For arbitrary k we then obtain the desired pullback diagram by taking the product of k copies of the $k = 1$ diagram over the $k = 0$ diagram. In particular, the desired pullback is given by the k -fold cartesian product $(T_x C)^k$, which exists since \mathbb{X} is a cartesian tangent ∞ -category. That pullback is preserved by $T^{A'}$ because this is true for each of the cases $k = 0, 1$, and because $T^{A'}$ preserves finite products, again since the tangent structure on \mathbb{X} is cartesian. \square

We now wish to take the pullback of the diagram 5.19 to obtain a tangent functor $\mathbb{E}_\infty \rightarrow \mathbb{X}$ whose underlying functor is given by $J \mapsto (T_x C)^J$. For this we need to choose a model of the pullback construction which is itself a tangent functor

$$\underline{\lim} : \mathbb{Y} \rightarrow \mathbb{X}.$$

We postpone the construction of $\underline{\lim}$ to the end of this section (Definition 5.30), but its existence should make sense given the definition of \mathbb{Y} in 5.22. Based on that construction we can now deduce the main result of this section.

PROPOSITION 5.24. *Let \mathbb{X} be a cartesian tangent ∞ -category. Then there is a functor*

$$\mathcal{J}_\bullet : \mathbb{X}_*^T \rightarrow \mathbb{D}\text{iff}(\mathbb{X})$$

*which maps a pointed object $x : * \rightarrow C$ that admits a tangent space $T_x C$ to a differential structure on that tangent space.*

PROOF. Composing the functor \mathcal{J}_\bullet of Lemma 5.23 with the map induced by the pullback tangent functor $\underline{\lim} : \mathbb{Y} \rightarrow \mathbb{X}$ of Definition 5.30 below we get a functor

$$\mathcal{J}_\bullet : \mathbb{X}_*^T \rightarrow \text{Fun}_{\text{lax}}^{\text{tan}}(\mathbb{E}_\infty, \mathbb{X})$$

which sends a pointed object $x : * \rightarrow C$ to a lax tangent functor $\mathcal{J}_x C : \mathbb{E}_\infty \rightarrow \mathbb{X}$ which, when evaluated at the object $*$ of \mathbb{E}_∞ , yields the tangent space $T_x C$. It only remains therefore to show that $\mathcal{J}_x C$ is, in fact, a strong tangent functor, and that its underlying functor $\mathbb{E}_\infty \rightarrow \mathbb{X}$ preserves products. The latter claim follows from the proof of Lemma 5.23 where we showed that $\mathcal{J}_x C(J)$ is equivalent to $(T_x C)^J$ for each finite set J .

So we are left to show that the $\mathcal{T}_x C$ is a strong tangent functor, i.e. that the Weil-module map $\mathcal{T}_x C : B\mathbb{E}_\infty \rightarrow \mathbb{X}$ is a map of marked simplicial sets. Equivalently we have to show that for each Weil-algebra A , and each J in \mathbb{E}_∞ , the map

$$\mathcal{T}_x C(M_A \times J) \rightarrow T^A \mathcal{T}_x C(J)$$

associated to the 1-simplex $(1_{\mathbb{N}}, 1_A, 1_J)$ in $B\mathbb{E}_\infty$, is an equivalence in \mathbb{X} . Our construction identifies this map with the following zigzag:

$$\lim \left(\begin{array}{c} T_{M_A \times J} C \\ \downarrow p_C \\ C \\ \uparrow x \\ * \end{array} \right) \rightarrow \lim \left(\begin{array}{c} T^A T_J C \\ \downarrow T^A p_C \\ T^A(C) \\ \uparrow T^A x \\ T^A(*) \end{array} \right) \leftarrow T^A \lim \left(\begin{array}{c} T_J C \\ \downarrow p_C \\ C \\ \uparrow x \\ * \end{array} \right)$$

where the second map is the canonical map into the pullback, which is an equivalence since we have shown in Lemma 5.23 that this pullback is preserved by T^A . The first map in the zigzag above is the map on pullbacks induced by the following diagram

$$(5.25) \quad \begin{array}{ccc} T_{M_A \times J}(C) & \xrightarrow{\iota(A, J)} & T^A T_J(C) \\ p_C \downarrow & & \downarrow T^A(p_C) \\ C & \xrightarrow{\iota(A, \emptyset)} & T^A(C) \\ x \uparrow & & \uparrow T^A(x) \\ * & \xrightarrow{\iota(A, \emptyset)} & T^A(*) \end{array}$$

where $\iota(A, J)$ is determined by the labelled Weil-algebra morphism $\mathcal{W}(1_{\mathbb{N}}, 1_A, 1_J)$ of Definition 5.13, i.e. the span of finite partial commutative monoids

$$\begin{array}{ccc} & \mathcal{W}(M_A \times J) & \\ & \parallel & \searrow \iota_{(M_A, J)} \\ \mathcal{W}(M_A \times J) & & M_A \times \mathcal{W}(J) \end{array}$$

where $\iota_{(M_A, J)}$ is as in Definition 5.12.

Since the bottom map $* \rightarrow T^A(*)$ is an equivalence (as the tangent structure on \mathbb{X} is cartesian), it is sufficient to show that the top square in (5.25) is a pullback for all Weil-algebras A and finite sets J .

Consider first the case $A = W$ and $|J| = 1$. In that case, the top square of (5.25) is precisely the vertical lift pullback square associated to the tangent structure in \mathbb{X} . (This is the point at which the vertical lift axiom is used in the proof of Proposition 5.24.)

Note also that when $A = \mathbb{N}$, the top and bottom maps in that square are identities, and when $|J| = 0$, the left and right maps are identities, so in those cases the square is a pullback too.

Next consider the case $A = W^n$ and $|J| = 1$ for arbitrary $n \geq 0$. In that case, the relevant square is a pullback power of n copies of the case $A = W$, $|J| = 1$, over the case $A = \mathbb{N}$, $|J| = 1$, so is a pullback too.

Now look at the case $A = W^n$ and J arbitrary. Then the square is a pullback power of $|J|$ copies of the case $A = W^n$, $|J| = 1$, over the case $A = W^n$, $|J| = 0$, so is a pullback too. This claim uses the fact that the tangent structure on \mathbb{X} preserves the foundational pullbacks of Proposition 2.31.

Finally, we consider the case of arbitrary A and J . Since any A is a tensor product of Weil-algebras of the form W^n , it is sufficient by induction to show that the case $A = A' \otimes A''$ follows from the cases for A' and A'' . Recall that $M_{A' \otimes A''} = M_{A'} \times M_{A''}$ and consider the diagram

$$\begin{array}{ccccc}
 T_{M_{A'} \times M_{A''} \times J}(C) & \xrightarrow{\iota(A', M_{A''} \times J)} & T^{A'} T_{M_{A''} \times J}(C) & \xrightarrow{T^{A'} \iota(A'', J)} & T^{A'} T^{A''} T_J(C) \\
 \downarrow & & \downarrow & & \downarrow \\
 C & \xrightarrow{\iota(A', \emptyset)} & T^{A'}(C) & \xrightarrow{T^{A'} \iota(A'', \emptyset)} & T^{A'} T^{A''}(C)
 \end{array}$$

The left-hand square is a pullback by hypothesis since it is the case for A' applied with J replaced by $M_{A''} \times J$.

For the right-hand square, we use the fact that the monoidal structure on $\mathbb{W}\text{eil}$ is symmetric: there is a canonical isomorphism $A' \otimes A'' \cong A'' \otimes A'$ given by matching up generators in the canonical way. Thus the right-hand square is equivalent in \mathbb{X} to the square for the case A'' applied to the object $T^{A'}(C)$, so is also a pullback by hypothesis.

Thus the larger square is a pullback too, which completes our proof that the top square in (5.25) is a pullback, and hence that $\mathcal{T}_x C$ is a strong tangent functor. Thus $\mathcal{T}_x C$ is a differential structure on $T_x C$ as claimed. \square

COROLLARY 5.26. *Let \mathbb{X} be a cartesian tangent ∞ -category. Then an object D in \mathbb{X} admits a differential structure if and only if D is equivalent to some tangent space $T_x C$.*

PROOF. The if direction follows from Proposition 5.24. To see the only if direction, it is sufficient to note that if D admits a differential structure, then we

have a diagram

$$\begin{array}{ccc}
 D & \xrightarrow{\langle \zeta, 1 \rangle} & D \times D \\
 \sim \downarrow & & \sim \downarrow \alpha_N \\
 T_\zeta D & \longrightarrow & T(D) \\
 \downarrow & & \downarrow p \\
 * & \xrightarrow{\zeta} & D
 \end{array}$$

and the top-left map is an equivalence because the bottom square and composite squares are both pullbacks in \mathbb{X} . \square

REMARK 5.27. Proposition 5.24 implies that a morphism $f : C \rightarrow D$ in \mathbb{X} induces a map

$$T_x f : T_x C \rightarrow T_{f(x)} D$$

that preserves the differential structures on these tangent spaces. This map $T_x f$ is the analogue of the ordinary derivative of a map f at a point x and preservation of the differential structure is the analogue of this derivative being a linear map in a setting where there is no precise version of the vector space structure on an ordinary tangent space.

Tangent ∞ -categories and cartesian differential categories. One of the motivations for Cockett and Cruttwell to study differential objects was to make a connection between cartesian tangent categories and the cartesian differential categories of Blute, Cockett and Seely [BCS09]. Roughly speaking, they show that for a cartesian tangent category \mathbb{X} in which every object has a canonical differential structure there is a corresponding cartesian differential structure on \mathbb{X} , and that every cartesian differential category arises in this way. We provide a generalization of that result to tangent ∞ -categories.

DEFINITION 5.28. Let \mathbb{X} be a cartesian tangent ∞ -category. We define a category $\widehat{h\text{Diff}}(\mathbb{X})$ with objects the differential objects of \mathbb{X} , and with morphisms from \mathcal{D} to \mathcal{D}' given by morphisms in the homotopy category of \mathbb{X} between the underlying objects $\mathcal{D}(\ast)$ and $\mathcal{D}'(\ast)$. This construction is not the homotopy category of $\text{Diff}(\mathbb{X})$ because we are now, like Cockett and Cruttwell, including *all* morphisms between the underlying objects, not only those that commute with the differential structures.

A full definition of *cartesian differential structure* on a category \mathbb{X} can be found in [BCS09, 2.1.1]. Such a structure consists of an assignment, to each morphism $f : A \rightarrow B$ in \mathbb{X} , a morphism $D(f) : A \times A \rightarrow B$, referred to as the *derivative* of f , satisfying a list of seven axioms.

The canonical example for the category whose objects are the Euclidean spaces \mathbb{R}^n , and whose morphisms are the smooth maps $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$, has derivative $\nabla(f) : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ given by the map

$$\nabla(f)(a, v) = D_a f(v)$$

where $D_a f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ denotes the derivative of f at the point $a \in \mathbb{R}^m$. In this example, the seven axioms describe standard properties of multivariable calculus such as the linearity of the total derivative and the equality of mixed partial derivatives.

The following result generalizes [CC14, 4.11].

THEOREM 5.29. *Let \mathbb{X} be a cartesian tangent ∞ -category. There is a cartesian differential structure on $\widehat{h\mathbb{D}\text{iff}}(\mathbb{X})$ in which the monoid structure on an object is that inherited from its differential structure, and the derivative of a morphism $f : A \rightarrow B$ is given by the composite*

$$\nabla(f) : A \times A \xrightarrow[\sim]{\langle \hat{p}_A, \hat{p}_A \rangle^{-1}} TA \xrightarrow{T(f)} TB \xrightarrow{\hat{p}_B} B.$$

where \hat{p}_A and \hat{p}_B are determined by the differential structures on A and B respectively.

PROOF. Let $\mathbb{X}_d \subseteq \mathbb{X}$ be the full subcategory of \mathbb{X} whose objects are those that admit a differential structure. We claim that \mathbb{X}_d is a tangent subcategory, i.e. is closed under the action of $\mathbb{W}\text{eil}$. To see this claim, we note that, if D admits a differential structure, $T^A(D) \simeq D^{M_A}$. A finite product of differential objects has a canonical differential structure, so \mathbb{X}_d is closed under finite products, and it follows that \mathbb{X}_d is a cartesian tangent subcategory of \mathbb{X} .

We now claim that, unusually, the homotopy category $h\mathbb{X}_d$ inherits a tangent structure from that on \mathbb{X}_d . The $\mathbb{W}\text{eil}$ -action on \mathbb{X}_d passes to an action of $\mathbb{W}\text{eil}_1$ on $h\mathbb{X}_d$, so the key is to show that each of the foundational and vertical lift pullbacks in \mathbb{X}_d is also a pullback in $h\mathbb{X}_d$.

First consider an object $D \in \mathbb{X}_d$ and finite sets J, J' . We have to show that

$$\begin{array}{ccc} T^{W^{J \sqcup J'}}(D) & \longrightarrow & T^{W^J}(D) \\ \downarrow & & \downarrow \\ T^{W^{J'}}(D) & \longrightarrow & D \end{array}$$

is a pullback in $h\mathbb{X}_d$. A differential structure on D determines equivalences in the ∞ -category \mathbb{X}_d , and hence isomorphisms in $h\mathbb{X}_f$ of the form

$$T^A(D) = T^A \mathcal{D}(\ast) \simeq \mathcal{D}(A \otimes \ast) = \mathcal{D}(M_A) \simeq \mathcal{D}(\ast)^{M_A} = D^{M_A}$$

where the penultimate equivalence comes from the fact that \mathcal{D} preserves finite products, which are given in \mathbb{E}_∞ by the disjoint union of sets. Note also that products in an ∞ -category pass to the homotopy category. The above equivalences are natural in A , which implies that we can replace the square above with a corresponding diagram

$$\begin{array}{ccc} D^{J \sqcup J' \sqcup \{1\}} & \longrightarrow & D^{J \sqcup \{1\}} \\ \downarrow & & \downarrow \\ D^{J' \sqcup \{1\}} & \longrightarrow & D \end{array}$$

where each map is the evident projection. A square of this form is a pullback in any category with finite products.

For the vertical lift axiom, we have to consider the following diagram in $h\mathbb{X}_d$

$$\begin{array}{ccc} D^3 & \xrightarrow{\langle \pi_1, \pi_2, \zeta^!, \pi_3 \rangle} & D^4 \\ \pi_3 \downarrow & & \downarrow \langle \pi_3, \pi_4 \rangle \\ D & \xrightarrow{\langle \zeta^!, 1 \rangle} & D^2 \end{array}$$

where $\zeta : * \rightarrow D$ is the zero map for the differential structure on D . A diagram of this form is a pullback in any category (regardless of the map ζ).

We have completed the check that $h\mathbb{X}_d$ has a cartesian tangent structure, and we now apply [CC14, 4.11] to this structure. This provides a cartesian differential structure on the category $\widehat{\mathbb{D}\text{iff}}(h\mathbb{X}_d)$ whose objects are the differential objects in $h\mathbb{X}_d$, and whose morphisms are morphisms in $h\mathbb{X}_d$ between the underlying objects.

We complete the proof of our theorem by constructing an embedding of the category $\widehat{h\text{Diff}}(\mathbb{X})$ into $\widehat{\mathbb{D}\text{iff}}(h\mathbb{X}_d)$. Given a differential object in \mathbb{X} with underlying object D , we obtain a differential structure on D in $h\mathbb{X}_d$ in the same manner as in the proof of Proposition 5.8. Since morphisms in each case are homotopy classes of maps in \mathbb{X} , we see that this construction determines a fully faithful functor

$$V : \widehat{h\text{Diff}}(\mathbb{X}) \rightarrow \widehat{\mathbb{D}\text{iff}}(h\mathbb{X}_d).$$

Since $\widehat{h\text{Diff}}(\mathbb{X})$ is closed under finite products, we obtain the desired cartesian differential structure by restriction along V . The formula for the derivative $\nabla(f)$ follows from that in $\widehat{\mathbb{D}\text{iff}}(h\mathbb{X}_d)$. \square

The one outstanding task in this chapter is to construct the pullback tangent functor $\underline{\text{lim}} : \mathbb{Y} \rightarrow \mathbb{X}$ used in the proof of Proposition 5.24. Recall that \mathbb{X} is a cartesian tangent ∞ -category and $\mathbb{Y} \subseteq \text{Fun}(\triangleleft, \mathbb{X})$ is a full subcategory of cospans each of which admits a pullback in \mathbb{X} that is preserved by the tangent structure map T^A for all Weil-algebras A .

DEFINITION 5.30. By definition of \mathbb{Y} we can choose a functor

$$\text{lim} : \mathbb{Y} \rightarrow \mathbb{X}$$

that calculates the pullback of each object in \mathbb{Y} . We first define a Weil-module map

$$\underline{\text{lim}}_0 : B_0\mathbb{Y} = \text{Weil} \times \mathbb{Y} \rightarrow \mathbb{X}$$

by freely extending lim , i.e.

$$\underline{\text{lim}}_0(A, D) := T^A(\text{lim } D).$$

The universal property of pullbacks then ensures that there is a functor

$$\text{Weil} \times \mathbb{Y} \rightarrow \text{Fun}(\Delta^1, \mathbb{X})$$

given by

$$(A, D) \mapsto [T^A(\text{lim } D) \rightarrow \text{lim } T^A D]$$

and our hypothesis on \mathbb{Y} is that this map is a natural equivalence. We can therefore choose a natural inverse, and extend freely, to obtain a Weil-module map

$$\underline{\text{lim}}_1 : B_1\mathbb{Y} = \text{Weil} \times \text{Weil} \times \mathbb{Y} \rightarrow \text{Fun}(\Delta^1, \mathbb{X})$$

given by equivalences

$$\underline{\lim}_1(A_1, A_0, D) = [T^{A_1} \lim T^{A_0} D \xrightarrow{\sim} T^{A_1} T^{A_0} \lim D].$$

Similarly, the universal property of pullbacks in an ∞ -category ensures that there is a functor

$$\mathbb{W}eil \times \mathbb{W}eil \times \mathbb{Y} \rightarrow \text{Fun}(\Delta^2, \mathbb{X})$$

consisting of 2-simplexes in \mathbb{X} of the form

$$\begin{array}{ccc} T^{A_1} T^{A_0} \lim D & \xrightarrow{\sim} & \lim T^{A_1} T^{A_0} D \\ & \searrow \sim & \nearrow \sim \\ & T^{A_1} \lim T^{A_0} D & \end{array}$$

whose edges are the functors whose inverses were used to construct $\underline{\lim}_1$. We have already chosen inverses for the edges of these 2-simplexes, and we can use the fact that $\text{Fun}(\mathbb{W}eil^2 \times \mathbb{Y}, \mathbb{X})$ is an ∞ -category to ‘fill in’ a corresponding ‘inverse’ for the 2-simplexes themselves. Extending freely, we obtain a $\mathbb{W}eil$ -module map

$$\underline{\lim}_2 : B_2 \mathbb{Y} = \mathbb{W}eil \times \mathbb{W}eil^2 \times \mathbb{Y} \rightarrow \text{Fun}(\Delta^2, \mathbb{X})$$

that is compatible with $\underline{\lim}_1$ via the face maps $B_2 \mathbb{Y} \rightarrow B_1 \mathbb{Y}$, and coface maps $\Delta^1 \rightarrow \Delta^2$.

Inductively, using the universal property of the ∞ -categorical pullback, we construct a sequence of functors

$$\underline{\lim}_n : B_n \mathbb{Y} \rightarrow \text{Fun}(\Delta^n, \mathbb{X})$$

which are compatible with the face maps on each side, i.e. we have a map of semi-simplicial objects

$$\underline{\lim}_\bullet : B_\bullet \mathbb{Y} \rightarrow \text{Fun}(\Delta^\bullet, \mathbb{X}).$$

This map induces a map

$$\underline{\lim}' : B' \mathbb{Y} \rightarrow \mathbb{X}$$

where $B' \mathbb{Y}$ denotes the so-called ‘fat’ geometric realization [Rie14, 8.5.12] of the simplicial object $B_\bullet \mathbb{Y}$. Since that fat realization is equivalent to the ordinary realization $B \mathbb{Y}$, and $B \mathbb{Y}$ is cofibrant as a marked $\mathbb{W}eil$ -module, we can lift $\underline{\lim}'$ to the desired map of $\mathbb{W}eil$ -modules

$$\underline{\lim} : B \mathbb{Y} \rightarrow \mathbb{X}$$

whose underlying functor is \lim .

Tangent Structures in and on an $(\infty, 2)$ -Category

The goal of this chapter is to generalize our notion of ‘tangent ∞ -category’ to $(\infty, 2)$ -categories in two different ways. Firstly, we will observe that one can easily extend Definition 3.2 to a notion of *tangent object in* any $(\infty, 2)$ -category, and that this context provides the natural setting for such a notion. Secondly, we apply that general notion to give a definition of *tangent $(\infty, 2)$ -category*, i.e. a tangent structure **on** an $(\infty, 2)$ -category, or tangent object in the $(\infty, 2)$ -category of $(\infty, 2)$ -categories.

Models for $(\infty, 2)$ -categories. The basic intuition for an $(\infty, 2)$ -category is that it is a category enriched in $(\infty, 1)$ -categories, i.e. in ∞ -categories. The most convenient model in many cases is to take this intuition as a definition.

DEFINITION 6.1. Let \mathbf{Cat}_∞ be the (ordinary) category of ∞ -categories with monoidal structure given by cartesian product. A *\mathbf{Cat}_∞ -category* \mathbf{C} is a category enriched in \mathbf{Cat}_∞ . In particular, for any object $\mathbb{X} \in \mathbf{C}$, the enrichment provides a strict monoidal ∞ -category

$$\mathrm{End}_{\mathbf{C}}(\mathbb{X}) := \mathrm{Hom}_{\mathbf{C}}(\mathbb{X}, \mathbb{X})$$

which can be used (see Definition 6.10) to define a notion of tangent structure on the object \mathbb{X} .

EXAMPLE 6.2. Let \mathbf{Cat}_∞ be the \mathbf{Cat}_∞ -category consisting of \mathbf{Cat}_∞ itself with self-enrichment given by its closed monoidal structure, i.e. with mapping objects

$$\mathrm{Hom}_{\mathbf{Cat}_\infty}(\mathbb{X}, \mathbb{Y}) = \mathrm{Fun}(\mathbb{X}, \mathbb{Y}).$$

We refer to \mathbf{Cat}_∞ as the *$(\infty, 2)$ -category of ∞ -categories*.

The \mathbf{Cat}_∞ -categories are the fibrant objects in a model structure on *marked simplicial categories*, i.e. categories enriched in the category \mathbf{Set}_Δ^+ of marked simplicial sets. That model structure is constructed in [Lur09a, A.3.2] and generalizes the Bergner model structure [Ber07] on simplicial categories. In [Lur09b], Lurie shows that this model structure is Quillen equivalent to other common models for $(\infty, 2)$ -categories.

From our perspective, the biggest drawback of using \mathbf{Cat}_∞ -categories to model $(\infty, 2)$ -categories is that they do not accurately describe the ‘mapping $(\infty, 2)$ -category’ between two $(\infty, 2)$ -categories. Given \mathbf{Cat}_∞ -categories \mathbf{C}, \mathbf{D} , one can define a new \mathbf{Cat}_∞ -category $\mathrm{Fun}_{\mathbf{Cat}_\infty}(\mathbf{C}, \mathbf{D})$ whose objects are the \mathbf{Cat}_∞ -enriched functors $\mathbf{C} \rightarrow \mathbf{D}$, yet this construction does not typically describe the correct ‘mapping $(\infty, 2)$ -category’, even when \mathbf{C} is cofibrant.¹

¹One explanation for this deficit is that the model structure on marked simplicial categories is not compatible with the cartesian monoidal structure.

It is therefore convenient to introduce a second model for $(\infty, 2)$ -categories for which these mapping $(\infty, 2)$ -categories are easier to describe; we will use Lurie's ∞ -bicategories from [Lur09b, 4.1] based on his notion of *scaled simplicial set* which we now recall.

DEFINITION 6.3. A *scaled simplicial set* (X, S) consists of a simplicial set X and a subset $S \subseteq X_2$ of the set of 2-simplexes in X , such that S contains all the degenerate 2-simplexes. We refer to the 2-simplexes in S as being *thin*. A *scaled morphism* $(X, S) \rightarrow (X', S')$ is a map of simplicial sets $f : X \rightarrow X'$ such that $f(S) \subseteq S'$. Let $\text{Set}_{\Delta}^{\text{sc}}$ denote the category of scaled simplicial sets and scaled morphisms.

There is a model structure on $\text{Set}_{\Delta}^{\text{sc}}$ described in [Lur09b, 4.2.7] which we take as our model for the ∞ -category of $(\infty, 2)$ -categories and refer to as the *scaled model structure*. The cofibrations in $\text{Set}_{\Delta}^{\text{sc}}$ are the monomorphisms, so every object is cofibrant. An ∞ -bicategory is a fibrant object in this model structure.

EXAMPLE 6.4. The Duskin nerve [Dus01] of an ordinary bicategory can be given the structure of an ∞ -bicategory; for example, see [Gur09, 3.16].

EXAMPLE 6.5. Given a Cat_{∞} -category \mathbf{C} , we define a scaled simplicial set, called the *scaled nerve* of \mathbf{C} , as follows. The underlying simplicial set of the scaled nerve is the simplicial nerve of the simplicially-enriched category \mathbf{C} ; see [Lur09a, 1.1.5.5]. A 2-simplex in that simplicial nerve is given by a (not-necessarily-commutative) diagram in \mathbf{C} of the form

$$\begin{array}{ccc} \mathbb{X}_0 & \xrightarrow{H} & \mathbb{X}_2 \\ & \searrow F & \nearrow G \\ & & \mathbb{X}_1 \end{array}$$

together with a morphism $\alpha : H \rightarrow GF$ in the ∞ -category $\text{Hom}_{\mathbf{C}}(\mathbb{X}_0, \mathbb{X}_2)$. We designate such a 2-simplex as *thin* if the corresponding morphism α is an equivalence. This choice of thin 2-simplexes gives the simplicial nerve of \mathbf{C} a scaling making it an ∞ -bicategory by [Lur09b, 4.2.7].

Lurie proves in [Lur09b, 4.2.7] that the scaled nerve is the right adjoint in a Quillen equivalence between the scaled model structure on $\text{Set}_{\Delta}^{\text{sc}}$ and the model structure on marked simplicial categories described above. In particular every ∞ -bicategory \mathbf{C} is weakly equivalent to the scaled nerve of a Cat_{∞} -category. When working with an arbitrary ∞ -bicategory, we will usually assume an implicit choice of such a Cat_{∞} -category, and hence of the corresponding mapping ∞ -categories $\text{Hom}_{\mathbf{C}}(\mathbb{X}, \mathbb{Y})$ for objects $\mathbb{X}, \mathbb{Y} \in \mathbf{C}$. Similarly, we will usually not distinguish notationally between the Cat_{∞} -category \mathbf{C} and the ∞ -bicategory given by its scaled nerve, extending our convention for nerves of categories.

EXAMPLE 6.6. Let \mathbf{C} be an ∞ -category. The *maximal scaling* on \mathbf{C} is that in which every 2-simplex is thin. This scaling makes \mathbf{C} into an ∞ -bicategory by [Lur09b, 4.1.2] and the fact, proved by Gagna, Harpaz and Lanari in [GHL, 5.1], that any weak ∞ -bicategory is also an ∞ -bicategory. In this way we treat an ∞ -category as a special kind of ∞ -bicategory.

Conversely, any ∞ -bicategory has an *underlying* ∞ -category.

DEFINITION 6.7. Let \mathbf{C} be an ∞ -bicategory, and denote by \mathbf{C}^{\simeq} the simplicial subset of \mathbf{C} consisting of those simplexes for which all of the 2-dimensional faces are

thin. Then \mathbf{C}^{\simeq} is an ∞ -category; this follows from [Lur09b, 4.1.3] and the fact that any scaled anodyne map is a trivial cofibration in $\mathbf{Set}_{\Delta}^{\text{sc}}$ (so that any ∞ -bicategory is also a ‘weak’ ∞ -bicategory).

We now describe the internal mapping objects for ∞ -bicategories.

DEFINITION 6.8. Let \mathbf{B} and \mathbf{C} be ∞ -bicategories. The ∞ -bicategory of functors $\mathbf{B} \rightarrow \mathbf{C}$ is the scaled simplicial set

$$\text{Fun}_{(\infty, 2)}(\mathbf{B}, \mathbf{C})$$

whose underlying simplicial set is the maximal simplicial subset of $\text{Fun}(\mathbf{B}, \mathbf{C})$ with vertices those maps $\mathbf{B} \rightarrow \mathbf{C}$ that are scaled morphisms. The scaling is that in which a 2-simplex $\Delta^2 \times \mathbf{B} \rightarrow \mathbf{C}$ is thin if it is a scaled morphism (where Δ^2 has the maximal scaling).

LEMMA 6.9. Let \mathbf{B}, \mathbf{C} be ∞ -bicategories. Then $\text{Fun}_{(\infty, 2)}(\mathbf{B}, \mathbf{C})$ is an ∞ -bicategory.

PROOF. Devalapurkar proves in [Dev16, 2.1] that $\text{Fun}_{(\infty, 2)}(-, -)$ makes $\mathbf{Set}_{\Delta}^{\text{sc}}$ into a cartesian closed model category. Since any object in $\mathbf{Set}_{\Delta}^{\text{sc}}$ is cofibrant, and \mathbf{B} is fibrant, it follows that $\text{Fun}_{(\infty, 2)}(\mathbf{B}, \mathbf{C})$ is fibrant. \square

Tangent objects in an $(\infty, 2)$ -category. We now turn to tangent structures, and our first goal is to describe what is meant by a tangent structure on an object *in* an $(\infty, 2)$ -category.

Let \mathbf{C} be a Cat_{∞} -category. Then any object $\mathbb{X} \in \mathbf{C}$ has a strict monoidal ∞ -category of ‘endofunctors’

$$\text{End}_{\mathbf{C}}(\mathbb{X}) := \text{Hom}_{\mathbf{C}}(\mathbb{X}, \mathbb{X})$$

with monoidal structure given by composition, and we therefore have the following simple generalization of Definition 3.2.

DEFINITION 6.10. Let \mathbf{C} be a Cat_{∞} -category, and let $\mathbb{X} \in \mathbf{C}$ be an object. A *tangent structure* on \mathbb{X} is a strict monoidal functor

$$T : \mathbf{Weil}^{\otimes} \rightarrow \text{End}_{\mathbf{C}}(\mathbb{X})^{\circ}$$

for which the underlying functor of ∞ -categories $\mathbf{Weil} \rightarrow \text{End}_{\mathbf{C}}(\mathbb{X})$ preserves the tangent pullbacks.

We can phrase Definition 6.10 in terms of functors of $(\infty, 2)$ -categories as follows. The strict monoidal ∞ -category \mathbf{Weil} determines a Cat_{∞} -category \mathbf{Weil} that has a single object \bullet , with $\text{Hom}_{\mathbf{Weil}}(\bullet, \bullet) := \mathbf{Weil}$. If we consider the identity morphisms in \mathbf{Weil} as marked, then it follows from Proposition 3.19 that \mathbf{Weil} is cofibrant in the model structure on marked simplicial categories. We make the following definition.

DEFINITION 6.11. Let \mathbf{C} be a Cat_{∞} -category. A *tangent object* in \mathbf{C} is a Cat_{∞} -enriched functor

$$\mathbf{T} : \mathbf{Weil} \rightarrow \mathbf{C}$$

for which the induced map on endomorphism ∞ -categories preserves the tangent pullbacks.

EXAMPLE 6.12. Let \mathbf{Cat}_{∞} denote the $(\infty, 2)$ -category of ∞ -categories of Example 6.2. Then a tangent object in \mathbf{Cat}_{∞} is a tangent ∞ -category in the sense of Definition 3.2.

Morphisms between tangent objects \mathbb{X} and \mathbb{Y} in an $(\infty, 2)$ -category \mathbf{C} can be defined via a generalization of the characterization of Lemma 4.11, with $\text{Fun}(\mathbb{X}, \mathbb{Y})$ replaced by the mapping object $\text{Hom}_{\mathbf{C}}(\mathbb{X}, \mathbb{Y})$.

However, we can now go further and define an entire $(\infty, 2)$ -category of tangent objects in \mathbf{C} . The following definition can be viewed as the culmination of the first half of this paper. Recall that we treat the Cat_{∞} -category \mathbf{Weil} as an ∞ -bicategory via the scaled nerve.

DEFINITION 6.13. Let \mathbf{C} be an ∞ -bicategory. The ∞ -bicategory of tangent objects in \mathbf{C} is the full $(\infty, 2)$ -subcategory (i.e. the maximal scaled simplicial subset)

$$\mathbf{Tan}(\mathbf{C}) \subseteq \text{Fun}_{(\infty, 2)}(\mathbf{Weil}, \mathbf{C})$$

whose vertices are those functors $\mathbf{T} : \mathbf{Weil} \rightarrow \mathbf{C}$ for which the induced functor on mapping ∞ -categories $\text{Weil} \rightarrow \text{End}_{\mathbf{C}}(\mathbb{X})$ preserves the tangent pullbacks.

REMARK 6.14. The mapping ∞ -categories in $\mathbf{Tan}(\mathbf{C})$ are generalizations of the ∞ -categories of strong tangent functors in Definition 4.9 and reduce, up to equivalence, to those ∞ -categories when $\mathbf{C} = \mathbf{Cat}_{\infty}$. Using work of Johnson-Freyd and Scheimbauer [JFS17] on lax natural transformations for (∞, n) -categories, we can also extend $\mathbf{Tan}(\mathbf{C})$ to an ∞ -bicategory of tangent objects and their *lax* tangent morphisms, though we will not pursue that extension here.

Tangent $(\infty, 2)$ -categories. The preceding theory defines a notion of tangent structure *in* an $(\infty, 2)$ -category, but it also determines a notion of tangent structure *on* an $(\infty, 2)$ -category, by applying Definition 6.10 in the case that $\mathbf{C} = \mathbf{Cat}_{(\infty, 2)}$ is a suitable $(\infty, 2)$ -category of $(\infty, 2)$ -categories.

DEFINITION 6.15. Let $\mathbf{Cat}_{(\infty, 2)}$ be the Cat_{∞} -category whose objects are the ∞ -bicategories, and whose mapping objects are the underlying ∞ -categories

$$\text{Hom}_{\mathbf{Cat}_{(\infty, 2)}}(\mathbb{X}, \mathbb{Y}) := \text{Fun}_{(\infty, 2)}(\mathbb{X}, \mathbb{Y})^{\simeq}$$

of the ∞ -bicategories of functors $\mathbb{X} \rightarrow \mathbb{Y}$. (See Definitions 6.7 and 6.8.)

An ∞ -bicategory \mathbb{X} has an endomorphism ∞ -category

$$\text{End}_{(\infty, 2)}(\mathbb{X}) := \text{Fun}_{(\infty, 2)}(\mathbb{X}, \mathbb{X})^{\simeq}$$

which is a strict monoidal ∞ -category under composition. Definition 6.10 then gives us the following notion.

DEFINITION 6.16. Let \mathbb{X} be an ∞ -bicategory. A *tangent structure* on \mathbb{X} is a strict monoidal functor

$$T : \text{Weil} \rightarrow \text{End}_{(\infty, 2)}(\mathbb{X})$$

for which the underlying functor $\text{Weil} \rightarrow \text{End}_{(\infty, 2)}(\mathbb{X})$ preserves tangent pullbacks. A *tangent ∞ -bicategory* consists of an ∞ -bicategory \mathbb{X} and a tangent structure T on \mathbb{X} .

The theory of tangent ∞ -bicategories subsumes that of tangent ∞ -categories: recall that an ∞ -category can be viewed as an ∞ -bicategory in which all 2-simplexes are thin.

LEMMA 6.17. *Let \mathbb{X} be an ∞ -category. Then a tangent structure on \mathbb{X} (in the sense of Definition 3.2) is the same thing as a tangent structure on the corresponding ∞ -bicategory \mathbb{X} (in the sense of Definition 6.16).*

PROOF. When \mathbb{X} is an ∞ -category, there is an isomorphism of strict monoidal ∞ -categories

$$\mathrm{End}_{(\infty, 2)}(\mathbb{X}) = \mathrm{Fun}_{(\infty, 2)}(\mathbb{X}, \mathbb{X})^{\simeq} = \mathrm{Fun}(\mathbb{X}, \mathbb{X}) = \mathrm{End}(\mathbb{X}).$$

□

REMARK 6.18. As with tangent ∞ -categories, we can also view a tangent structure on an ∞ -bicategory \mathbb{X} via an ‘action’ map:

$$T : \mathrm{Weil} \times \mathbb{X} \rightarrow \mathbb{X}$$

which is an action of the scaled simplicial monoid Weil (where every 2-simplex is thin) on the scaled simplicial set \mathbb{X} .

Unlike the situation for a tangent structure on an ∞ -category, however, we do not know a simple characterization of when an action map of the form T corresponds to a map of ∞ -categories $\mathrm{Weil} \rightarrow \mathrm{End}_{(\infty, 2)}(\mathbb{X})$ which preserves the tangent pullbacks. The issue is that we don’t know in general how to identify pullbacks in an ∞ -category of the form $\mathrm{Fun}_{(\infty, 2)}(\mathbb{X}, \mathbb{Y})^{\simeq}$. Nonetheless we can provide a sufficient criterion for a diagram in this ∞ -category to be a pullback, and we will use this criterion in Chapter 12 to verify that the Goodwillie tangent structure extends to an ∞ -bicategory.

Our criterion is based on the following notion of pullback in an $(\infty, 2)$ -category. Recall that we define mapping ∞ -categories $\mathrm{Hom}_{\mathbb{X}}(C, D)$ by choosing a Cat_{∞} -category whose scaled nerve is equivalent to \mathbb{X} .

DEFINITION 6.19. Let \mathbb{X} be an ∞ -bicategory, and consider a commutative diagram (i.e. a pair of thin 2-simplexes with a common edge):

$$\begin{array}{ccc} C & \longrightarrow & C_1 \\ \downarrow & \searrow & \downarrow \\ C_2 & \longrightarrow & C_0 \end{array}$$

in \mathbb{X} . We say that this diagram is a *homotopy 2-pullback* if, for every $D \in \mathbb{X}$, the induced diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{X}}(D, C) & \longrightarrow & \mathrm{Hom}_{\mathbb{X}}(D, C_1) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathbb{X}}(D, C_2) & \longrightarrow & \mathrm{Hom}_{\mathbb{X}}(D, C_0) \end{array}$$

is a homotopy pullback of ∞ -categories (that is, a homotopy pullback in the Joyal model structure on simplicial sets).

PROPOSITION 6.20. *Let \mathbb{X} be an ∞ -bicategory, and let*

$$T : \mathrm{Weil} \times \mathbb{X} \rightarrow \mathbb{X}$$

be an action of the simplicial monoid Weil (with every 2-simplex considered thin) on the scaled simplicial set \mathbb{X} . Suppose that for each of the tangent pullback squares

in Weil

$$\begin{array}{ccc} A & \longrightarrow & A_1 \\ \downarrow & \searrow & \downarrow \\ A_2 & \longrightarrow & A_0 \end{array}$$

and each object $C \in \mathbb{X}$, the resulting diagram

$$\begin{array}{ccc} T^A C & \longrightarrow & T^{A_1} C \\ \downarrow & \searrow & \downarrow \\ T^{A_2} C & \longrightarrow & T^{A_0} C \end{array}$$

is a homotopy 2-pullback in \mathbb{X} . Then T defines a tangent structure on \mathbb{X} .

PROOF. We need to show that each diagram

$$\begin{array}{ccc} T^A & \longrightarrow & T^{A_1} \\ \downarrow & \searrow & \downarrow \\ T^{A_2} & \longrightarrow & T^{A_0} \end{array}$$

determines a pullback in the ∞ -category $\text{End}_{(\infty, 2)}(\mathbb{X}) = \text{Fun}_{(\infty, 2)}(\mathbb{X}, \mathbb{X})^{\simeq}$. That claim follows from the following more general lemma. \square

LEMMA 6.21. *Let \mathbb{X}, \mathbb{Y} be ∞ -bicategories, and let $D : \square \times \mathbb{X} \rightarrow \mathbb{Y}$ be a scaled morphism (where every 2-simplex in $\square := \Delta^1 \times \Delta^1$ is considered thin) such that for each $x \in \mathbb{X}$, the resulting diagram $D(-, x) : \square \rightarrow \mathbb{Y}$ is a homotopy 2-pullback in \mathbb{Y} . Then D corresponds to a pullback square in the ∞ -category $\text{Fun}_{(\infty, 2)}(\mathbb{X}, \mathbb{Y})^{\simeq}$.*

PROOF. To get a handle on the ∞ -category $\text{Fun}_{(\infty, 2)}(\mathbb{X}, \mathbb{Y})^{\simeq}$ we translate this problem into the context of Cat_{∞} -categories and use the following characterization by Lurie [Lur09a, A.3.4] for the (derived) internal mapping object for Cat_{∞} -categories.

Without loss of generality, we may assume that \mathbb{X} and \mathbb{Y} are (the scaled nerves of) Cat_{∞} -categories. It is then possible to form the Cat_{∞} -category

$$\text{Fun}_{\text{Cat}_{\infty}}(\mathbb{X}, \mathbb{Y})$$

of Cat_{∞} -enriched functors $\mathbb{X} \rightarrow \mathbb{Y}$, but typically this Cat_{∞} -category does *not* model the $(\infty, 2)$ -category $\text{Fun}_{(\infty, 2)}(\mathbb{X}, \mathbb{Y})$. The problem, in some sense, is that there is no notion of when a functor $\mathbb{X} \rightarrow \mathbb{Y}$ is ‘cofibrant’. We rectify this concern by embedding \mathbb{Y} into a suitable model category via its Yoneda map.

The enriched Yoneda embedding for \mathbb{Y} determines a fully faithful map of Cat_{∞} -categories

$$\text{Fun}_{\text{Cat}_{\infty}}(\mathbb{X}, \mathbb{Y}) \rightarrow \text{Fun}_{\text{Cat}_{\infty}}(\mathbb{X} \times \mathbb{Y}^{op}, \text{Cat}_{\infty}).$$

The right-hand side here can be identified with the full subcategory of fibrant objects in the category

$$\text{Fun}_{\text{Set}_{\Delta}^+}(\mathbb{X} \times \mathbb{Y}^{op}, \text{Set}_{\Delta}^+)$$

of \mathbf{Set}_Δ^+ -enriched functors $\mathbb{X} \times \mathbb{Y}^{op} \rightarrow \mathbf{Set}_\Delta^+$ equipped with the projective model structure in which an \mathbf{Set}_Δ^+ -enriched natural transformation is a weak equivalence and/or fibration if and only if each of its components is a weak equivalence and/or fibration in the marked model structure on \mathbf{Set}_Δ^+ .

It follows from [Lur09a, A.3.4.14] that the correct internal mapping object for \mathbf{Cat}_∞ -categories is given by the full subcategory of $\mathbf{Fun}_{\mathbf{Cat}_\infty}(\mathbb{X} \times \mathbb{Y}^{op}, \mathbf{Cat}_\infty)$ whose objects are functors $F : \mathbb{X} \times \mathbb{Y}^{op} \rightarrow \mathbf{Cat}_\infty$ which are also cofibrant in the model structure described above, and which have the property that, for each $x \in \mathbb{X}$, the \mathbf{Cat}_∞ -enriched functor $F(x, -) : \mathbb{Y}^{op} \rightarrow \mathbf{Cat}_\infty$ is in the essential image of the Yoneda embedding. We denote that full subcategory by

$$\widetilde{\mathbf{Fun}}_{\mathbf{Cat}_\infty}(\mathbb{X}, \mathbb{Y}).$$

By [Lur09a, A.3.4.14], the diagram D corresponds to a homotopy coherent square \bar{D} in the \mathbf{Cat}_∞ -category $\widetilde{\mathbf{Fun}}_{\mathbf{Cat}_\infty}(\mathbb{X}, \mathbb{Y})$, and the hypothesis on D implies that for each $x \in \mathbb{X}$, the resulting square $\bar{D}(x, -) : \mathbb{Y}^{op} \rightarrow \mathbf{Cat}_\infty$ has the property that for each $y \in \mathbb{Y}$, the diagram $\bar{D}(x, y)$ is a homotopy pullback of ∞ -categories, i.e. a homotopy pullback in the marked model structure on \mathbf{Set}_Δ^+ . Since fibrations and weak equivalences are detected objectwise in the projective model structure on $\mathbf{Fun}_{\mathbf{Set}_\Delta^+}(\mathbb{X} \times \mathbb{Y}^{op}, \mathbf{Set}_\Delta^+)$, this condition implies that \bar{D} is a homotopy pullback square of fibrant-cofibrant objects in that model structure, and hence determines a pullback in the corresponding ∞ -category. Since $\mathbf{Fun}_{(\infty, 2)}(\mathbb{X}, \mathbb{Y})^\simeq$ is equivalent to a full subcategory of that ∞ -category, it follows that D is a pullback there too. \square

COROLLARY 6.22. *Let (\mathbb{X}, T) be a tangent ∞ -bicategory that satisfies the condition of Proposition 6.20. Then T restricts to a tangent structure on the underlying ∞ -category \mathbb{X}^\simeq .*

PROOF. Applying $(-)^{\simeq}$ to the map of ∞ -bicategories

$$T : \mathbf{Weil} \times \mathbb{X} \rightarrow \mathbb{X}$$

we get the desired action map

$$T^\simeq : \mathbf{Weil} \times \mathbb{X}^\simeq \rightarrow \mathbb{X}^\simeq.$$

A homotopy pullback of ∞ -categories (in the Joyal model structure) determines a homotopy pullback of maximal Kan-subcomplexes (in the Quillen model structure) and so a homotopy 2-pullback in \mathbb{X} determines a pullback in the ∞ -category \mathbb{X}^\simeq . Thus the condition of Proposition 6.20 verifies the necessary tangent pullbacks in \mathbb{X}^\simeq . \square

Part 2

A Tangent ∞ -Category of ∞ -Categories

Goodwillie Calculus and the Tangent Bundle Functor

Up to this point we have been developing the general theory of tangent ∞ -categories, but now we start to look at the specific tangent structure which motivated this work. The construction and study of this structure, which we term the *Goodwillie tangent structure*, occupies the remainder of this paper. We start with a brief review of Goodwillie’s calculus of functors, ideas from which will permeate the definitions and proofs to come. Most of this chapter is based on Lurie’s development [Lur17, Ch. 6] of Goodwillie’s ideas in the general context of ∞ -categories.

The central notion in Goodwillie calculus is the *Taylor tower* of a functor; that is, a sequence of approximations to the functor that plays the role of the Taylor series in ordinary calculus. To describe the Taylor tower, we recall Goodwillie’s analogues of polynomials in the context of functors. We need the following preliminary definition.

DEFINITION 7.1. An *n-cube* in an ∞ -category \mathcal{C} is a diagram \mathcal{X} in \mathcal{C} indexed by the poset $\mathcal{P}[n]$ of subsets of $[n] = \{1, \dots, n\}$. Such a cube is *strongly cocartesian* if each 2-dimensional face is a pushout, and is *cartesian* if the cube as a whole is a limit diagram, i.e. the map

$$\mathcal{X}(\emptyset) \rightarrow \operatorname{holim}_{\emptyset \neq S \subseteq [n]} \mathcal{X}(S)$$

is an equivalence in \mathcal{C} .

DEFINITION 7.2 (Goodwillie [Goo91, 3.1]). Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between ∞ -categories. We say that F is *n-excisive* if it maps each strongly cocartesian $(n + 1)$ -cube in \mathcal{C} to a cartesian cube in \mathcal{D} . In particular, F is 1-excisive (or, simply, *excisive*) if it maps pushout squares in \mathcal{C} to pullback squares in \mathcal{D} , and F is 0-excisive if and only if it is constant (up to equivalence). We write

$$\operatorname{Exc}^n(\mathcal{C}, \mathcal{D}) \subseteq \operatorname{Fun}(\mathcal{C}, \mathcal{D})$$

for the full subcategory of the functor ∞ -category whose objects are the *n-excisive* functors.

REMARK 7.3. Motivation for this definition comes from algebraic topology; the property of being excisive is closely related to the excision property in the Eilenberg-Steenrod axioms for homology, and excisive functors on the ∞ -category of pointed topological spaces are closely related to homology theories.

One of Goodwillie’s key constructions is of a universal *n-excisive* approximation for functors between suitable ∞ -categories. In order to state a condition for this approximation to exist, we introduce the following definition from Lurie [Lur17, 6.1.1.6].

DEFINITION 7.4. An ∞ -category \mathcal{C} is *differentiable* if it admits finite limits and sequential colimits (i.e. colimits along countable sequences of composable morphisms), and those limits and colimits commute.

REMARK 7.5. We caution the reader to distinguish carefully between the words ‘differentiable’, as applied to an ∞ -category in the above definition, and ‘differential’, which we introduced in Chapter 5 to refer to the structure inherent on a tangent space in any tangent ∞ -category. In Chapter 10 we make that distinction particularly challenging by considering differential objects in the ∞ -category of differentiable ∞ -categories, or ‘differential differentiable ∞ -categories’ if you prefer.

We did consider introducing a new term for what Lurie calls ‘differentiable’ ∞ -categories in order to avoid the clash of terminology described above. For example, we might have called them ‘Goodwillie’ ∞ -categories in order to emphasize their role in the theory of Goodwillie calculus. They are also closely related to the ‘precontinuous categories’ of [ALR03, 1.2], and we might have introduced the term ‘finitely precontinuous ∞ -category’ for the notion relevant here.

Ultimately, we decided to adopt Lurie’s terminology despite the potential for confusion. While that terminology does not yet appear to be well-established across the literature, our work involves extensive references to [Lur17], where the term ‘differentiable’ was introduced and which itself is a widely-read work. We found it preferable to retain the same language, and to trust that the reader will carefully distinguish between the terms ‘differentiable’ and ‘differential’.

The following result is due to Lurie [Lur17, 6.1.1.10] in this generality, but its proof is based on Goodwillie’s original construction from [Goo03, Sec. 1].

PROPOSITION 7.6. *Let \mathcal{C}, \mathcal{D} be ∞ -categories such that \mathcal{C} has finite colimits and a terminal object, and \mathcal{D} is differentiable. Then the inclusion*

$$\mathrm{Exc}^n(\mathcal{C}, \mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

admits a left adjoint P_n which preserves finite limits.

Proposition 7.6 implies that a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ admits a universal n -excisive approximation map $p_n : F \rightarrow P_n F$, with the property that any map $F \rightarrow G$, where G is n -excisive, factors uniquely (i.e. up to contractible choice) as $F \rightarrow P_n F \rightarrow G$.

DEFINITION 7.7. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between ∞ -categories that satisfy the conditions in Proposition 7.6. The *Taylor tower* of F is the sequence of natural transformations

$$F \rightarrow \cdots \rightarrow P_n F \rightarrow P_{n-1} F \rightarrow \cdots \rightarrow P_1 F \rightarrow P_0 F = F(*)$$

determined by the universal property of each $P_n F$ and the observation that an n -excisive functor is also $(n+1)$ -excisive, for each n .

We need the following generalization of [AC11, 3.1].

LEMMA 7.8. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$ be functors between ∞ -categories that satisfy the conditions of Proposition 7.6, and suppose that F preserves the terminal object in \mathcal{C} . Then the map*

$$P_n(GF) \rightarrow P_n((P_n G)F)$$

induced by $p_n : G \rightarrow P_n G$, is an equivalence.

PROOF. Following Goodwillie [Goo03], Lurie defines [Lur17, 6.1.1.27]

$$P_n F := \operatorname{colim}_k T_n^k F$$

where

$$T_n F(X) \simeq \lim_{\emptyset \neq S \subseteq [n]} F(C_S(X))$$

and $C_S(X) \simeq \operatorname{hocofib}(\bigvee_S X \rightarrow X)$ is a model for the ‘join’ of the object $X \in \mathcal{C}$ with a finite set S . Since F preserves the terminal object, we get canonical maps

$$C_S(F(X)) \rightarrow F(C_S(X))$$

and hence

$$T_n(G)F \rightarrow T_n(GF)$$

and therefore also a map (commuting with the canonical maps from GF) of the form

$$P_n(G)F \rightarrow P_n(GF).$$

Since $P_n(GF)$ is n -excisive, this must factor via a map

$$P_n((P_n G)F) \rightarrow P_n(GF)$$

which we claim to be an inverse to the map in the statement of the Lemma. It follows from the universal property of P_n that the composite

$$P_n(GF) \rightarrow P_n((P_n G)F) \rightarrow P_n(GF)$$

is an equivalence. In the following diagram

$$\begin{array}{ccccc} P_n(GF) & \longrightarrow & P_n((P_n G)F) & \longrightarrow & P_n(GF) \\ \downarrow & & \downarrow & & \downarrow \\ P_n((P_n G)F) & \longrightarrow & P_n P_n((P_n G)F) & \longrightarrow & P_n((P_n G)F) \end{array}$$

the horizontal composites are equivalences, and the middle vertical map is an equivalence. It follows that the end map is a retract of an equivalence, hence is itself an equivalence. \square

A significant role in our later constructions is played by a multivariable version of Goodwillie’s calculus, so we describe that briefly too. More details are in [Lur17, 6.1.3].

DEFINITION 7.9. Let $F : \mathcal{C}_1 \times \mathcal{C}_2 \rightarrow \mathcal{D}$ be a functor between ∞ -categories such that each of $\mathcal{C}_1, \mathcal{C}_2$ admits finite colimits and a terminal object, and \mathcal{D} is differentiable. We say that F is n_1 -excisive in its first variable (and similarly for its second variable) if, for all $X_2 \in \mathcal{C}_2$, the functor $F(-, X_2) : \mathcal{C}_1 \rightarrow \mathcal{D}$ is n_1 -excisive. We say that F is (n_1, n_2) -excisive if it is n_i -excisive in its i -th variable, for each i . We write

$$\operatorname{Exc}^{n_1, n_2}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D}) \subseteq \operatorname{Fun}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D})$$

for the full subcategory on the (n_1, n_2) -excisive functors. The inclusion of this subcategory has a left adjoint P_{n_1, n_2} given by applying the functor P_{n_i} of Proposition 7.6 to each variable in turn, keeping the other variable constant.

We now begin our description of the Goodwillie tangent structure. First we introduce the underlying ∞ -category for this structure.

DEFINITION 7.10. Let $\mathbb{C}at_\infty$ be Lurie’s model [Lur09a, 3.0.0.1] for the ∞ -category of ∞ -categories¹, and let $\mathbb{C}at_\infty^{\text{diff}} \subseteq \mathbb{C}at_\infty$ be the subcategory whose objects are the differentiable ∞ -categories and whose morphisms are the functors that preserve sequential colimits.

The tangent bundle functor for the Goodwillie tangent structure is an endofunctor $T : \mathbb{C}at_\infty^{\text{diff}} \rightarrow \mathbb{C}at_\infty^{\text{diff}}$ defined by Lurie in [Lur17, 7.3.1.10]. That definition, and much of the rest of this paper, depends heavily on the ∞ -category $\mathcal{S}_{\text{fin},*}$ of ‘finite pointed spaces’, so let us be entirely explicit about that object.

DEFINITION 7.11. We say that a simplicial set is *finite* if it is homotopy-equivalent to the singular simplicial set on a finite CW-complex. Let $\mathcal{S}_{\text{fin},*}$ denote the simplicial nerve of the simplicial category in which an object is a pointed Kan complex (X, x_0) with X finite, with enrichment given by the pointed mapping spaces

$$\text{Hom}_{\mathcal{S}_{\text{fin},*}}((X, x_0), (Y, y_0)) \subseteq \text{Hom}(X, Y)$$

whose vertices are the basepoint-preserving maps $X \rightarrow Y$. Since these mapping spaces are Kan complexes, $\mathcal{S}_{\text{fin},*}$ is an ∞ -category by [Lur09a, 1.1.5.10].

Explicitly, an object of $\mathcal{S}_{\text{fin},*}$ is a finite pointed Kan complex, and a morphism is a basepoint-preserving map. A 2-simplex in $\mathcal{S}_{\text{fin},*}$ consists of a diagram of basepoint-preserving maps

$$\begin{array}{ccc} (X, x_0) & \xrightarrow{h} & (Z, z_0) \\ & \searrow f & \nearrow g \\ & & (Y, y_0) \end{array}$$

together with a basepoint-preserving simplicial homotopy $h \simeq gf$. Higher simplices in $\mathcal{S}_{\text{fin},*}$ involve more complicated diagrams of basepoint-preserving maps and higher-dimension homotopies.

DEFINITION 7.12 ([Lur17, 7.3.1.10]). Let \mathcal{C} be a differentiable ∞ -category. The *tangent bundle* on \mathcal{C} is the ∞ -category

$$T(\mathcal{C}) := \text{Exc}(\mathcal{S}_{\text{fin},*}, \mathcal{C})$$

of excisive functors $\mathcal{S}_{\text{fin},*} \rightarrow \mathcal{C}$ (i.e. those that map pushout squares in $\mathcal{S}_{\text{fin},*}$ to pullback squares in \mathcal{C}).

The tangent bundle on \mathcal{C} is equipped with a projection map

$$p_{\mathcal{C}} : T(\mathcal{C}) \rightarrow \mathcal{C}; \quad L \mapsto L(*)$$

given by evaluating an excisive functor L at the one-point space $*$.

REMARK 7.13. Lurie actually makes Definition 7.12 for a slightly different class of ∞ -categories: those that are *presentable* in the sense of [Lur09a, 5.5.0.1]. There is a significant overlap between the presentable and differentiable ∞ -categories including any compactly-generated ∞ -category and any ∞ -topos; see [Lur17, 6.1.1].

It seems very likely that most of the rest of this paper could be made with the ∞ -category $\mathbb{C}at_\infty^{\text{diff}}$ replaced by an ∞ -category $\mathbb{C}at_\infty^{\text{pres}}$ of presentable ∞ -categories in which the morphisms are those functors that preserve all *filtered* colimits, not

¹Recall that we are not being explicit about the various size restrictions on our ∞ -categories, but for this definition to make sense, we of course require the objects in the ∞ -category $\mathbb{C}at_\infty$ to be restricted to be smaller than $\mathbb{C}at_\infty$ itself.

merely the sequential colimits. For example, Proposition 7.6 holds when \mathcal{D} is a presentable ∞ -category, and [Lur17, 7.3.1.14] implies that Definition 7.14 allows for T to be extended to a functor $T : \text{Cat}_\infty^{\text{pres}} \rightarrow \text{Cat}_\infty^{\text{pres}}$. However, much of our argument is based on results from [Lur17, Sec. 6] which is written in the context of differentiable ∞ -categories, so we follow that lead.

DEFINITION 7.14. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between differentiable ∞ -categories. We define a functor

$$T(F) : T(\mathcal{C}) \rightarrow T(\mathcal{D})$$

by the formula

$$L \mapsto P_1(FL).$$

It follows from Lemma 8.10 below that the constructions of Definitions 7.12 and 7.14 can be made into a functor

$$T : \text{Cat}_\infty^{\text{diff}} \rightarrow \text{Cat}_\infty^{\text{diff}}$$

so that the projection maps p_e form a natural transformation p from T to the identity functor I .

The following result is the main theorem of this paper.

THEOREM 7.15. *The tangent bundle functor $T : \text{Cat}_\infty^{\text{diff}} \rightarrow \text{Cat}_\infty^{\text{diff}}$ and the projection map $p : T \rightarrow I$ extend to a tangent structure (in the sense of Definition 3.2) on the ∞ -category $\text{Cat}_\infty^{\text{diff}}$. We refer to this structure as the Goodwillie tangent structure.*

We believe that the tangent structure described in Theorem 7.15 is *unique* (i.e. that the space of tangent structures that extend T and p is contractible) though we will not prove that claim here.

The proof of Theorem 7.15 occupies the next two chapters, concluding with the proof of Theorem 9.22. In Chapter 8 we introduce the basic constructions and definitions which underlie the tangent structure; that is, we describe how to construct, from a Weil-algebra A and a differentiable ∞ -category \mathcal{C} , a new ∞ -category $T^A(\mathcal{C})$, and how these constructions interact with labelled morphisms of Weil-algebras and (sequential-colimit-preserving) functors of ∞ -categories.

In Chapter 9 we turn those constructions into an actual tangent structure on the ∞ -category $\text{Cat}_\infty^{\text{diff}}$. It turns out to be more convenient to define that tangent structure on an ∞ -category $\text{RelCat}_\infty^{\text{diff}}$ that is equivalent to $\text{Cat}_\infty^{\text{diff}}$, and whose definition is based on the notion of *relative* ∞ -category which we describe in Definition 9.3. Having established a tangent structure on $\text{RelCat}_\infty^{\text{diff}}$, we transfer that structure to $\text{Cat}_\infty^{\text{diff}}$ using Lemma 3.12.

In Chapter 10 we begin the study of the Goodwillie tangent structure by identifying its differential objects in the sense of Definition 5.7. Those differential objects turn out to be precisely the stable ∞ -categories. That fact is of no surprise given that the tangent bundle construction 7.12 is formed precisely so that its tangent spaces are the stable ∞ -categories $Sp(\mathcal{C}/_X)$. Nonetheless this observation is a check that our tangent structure is acting as intended.

The characterization of differential objects as stable ∞ -categories also confirms the intuition, promoted by Goodwillie, that in the analogy between functor calculus and the ordinary calculus of manifolds one should view the category of spectra as playing the role of Euclidean space.

Further developing that analogy, in Chapter 11 we characterize the *n-excisive* functors for $n > 1$, as corresponding to a notion of ‘*n*-jet’ of a smooth map between manifolds. The construction of a Taylor tower itself does not precisely fit into the framework of tangent ∞ -categories because it involves non-invertible natural transformations. We therefore show in Chapter 12 that the Goodwillie tangent structure on $\mathbb{C}at_{\infty}^{\text{diff}}$ extends to a tangent structure, in the sense of Definition 6.10, on an ∞ -bicategory $\mathbb{C}AT_{\infty}^{\text{diff}}$ whose underlying ∞ -category is $\mathbb{C}at_{\infty}^{\text{diff}}$. That tangent ∞ -bicategory is the natural setting for Goodwillie calculus.

The Goodwillie Tangent Structure: Underlying Data

Our goal in this chapter is to introduce the basic data of the Goodwillie tangent structure on the ∞ -category $\mathbb{C}\text{at}_{\infty}^{\text{diff}}$ without paying attention to the higher coherence information needed to obtain an actual tangent ∞ -category. Thus we define the tangent structure on objects and morphisms in $\mathbb{W}\text{eil}$ and $\mathbb{C}\text{at}_{\infty}^{\text{diff}}$, and prove basic lemmas concerning functoriality and the preservation of pullbacks, including the crucial vertical lift axiom (Proposition 8.36).

Tangent structure on objects. We start by defining $T^A(\mathcal{C})$ for a Weil-algebra A and a differentiable ∞ -category \mathcal{C} . When $A = W$, this definition reduces precisely to Definition 7.12.

DEFINITION 8.1. Let $A = W^{n_1} \otimes \cdots \otimes W^{n_r}$ be an object in the ∞ -category $\mathbb{W}\text{eil}$ with $n = n_1 + \cdots + n_r$ generators. We write

$$\mathcal{S}_{\text{fin},*}^n := \mathcal{S}_{\text{fin},*} \times \cdots \times \mathcal{S}_{\text{fin},*}$$

for the product of n copies of the ∞ -category $\mathcal{S}_{\text{fin},*}$ (with $\mathcal{S}_{\text{fin},*}^0 = *$). Let

$$(8.2) \quad T^A(\mathcal{C}) = \text{Exc}^A(\mathcal{S}_{\text{fin},*}^n, \mathcal{C}) := \text{Exc}^{1,\dots,1}(\mathcal{S}_{\text{fin},*}^{n_1} \times \cdots \times \mathcal{S}_{\text{fin},*}^{n_r}, \mathcal{C})$$

be the full subcategory of the functor ∞ -category $\text{Fun}(\mathcal{S}_{\text{fin},*}^n, \mathcal{C})$ that consists of those functors

$$L : \mathcal{S}_{\text{fin},*}^{n_1} \times \cdots \times \mathcal{S}_{\text{fin},*}^{n_r} \rightarrow \mathcal{C}$$

that are excisive (i.e. take pushouts to pullbacks) in each of their r variables individually; see Definition 7.9. We say that a functor L with this property is *A-excisive*.

It is crucial in Definition 8.1 that we view L as a functor of r variables, each of the form $\mathcal{S}_{\text{fin},*}^{n_j}$, according to the description of the Weil-algebra A as a tensor product of terms of the form W^{n_j} .

EXAMPLES 8.3. We can identify some particular examples of $T^A(\mathcal{C})$ to show that we are on the right track to define a tangent structure.

- (1) For $A = \mathbb{N}$, the unit object for the monoidal structure on $\mathbb{W}\text{eil}$, we get the identity functor:

$$T^{\mathbb{N}}(\mathcal{C}) = \text{Fun}(\mathcal{S}_{\text{fin},*}^0, \mathcal{C}) \cong \mathcal{C}.$$

- (2) For $A = W = \mathbb{N}[x]/(x^2)$, we get the tangent bundle functor from Definition 7.12:

$$T^W(\mathcal{C}) = T(\mathcal{C}) = \text{Exc}(\mathcal{S}_{\text{fin},*}, \mathcal{C}).$$

- (3) For $A = W \otimes W = \mathbb{N}[x, y]/(x^2, y^2)$, we get the ∞ -category of functors $\mathcal{S}_{\text{fin},*}^2 \rightarrow \mathcal{C}$ that are excisive *in each variable individually*. In the notation of [Lur17, 6.1.3.1] we can write

$$T^{W \otimes W}(\mathcal{C}) = \text{Exc}^{1,1}(\mathcal{S}_{\text{fin},*}^2, \mathcal{C}),$$

and we have $T^{W \otimes W}(\mathcal{C}) \simeq T^W(T^W(\mathcal{C}))$ as required in a tangent structure; see [Lur17, 6.1.3.3].

- (4) For $A = W^2 = \mathbb{N}[x, y]/(x^2, xy, y^2)$, we get the ∞ -category of functors $\mathcal{S}_{\text{fin},*}^2 \rightarrow \mathcal{C}$ that are excisive when viewed as a functor of *one* variable:

$$T_2(\mathcal{C}) := T^{W^2}(\mathcal{C}) = \text{Exc}(\mathcal{S}_{\text{fin},*}^2, \mathcal{C}).$$

For this definition to satisfy the pullback conditions in a tangent structure, we need to have an equivalence of ∞ -categories

$$\text{Exc}(\mathcal{S}_{\text{fin},*}^2, \mathcal{C}) \simeq \text{Exc}(\mathcal{S}_{\text{fin},*}, \mathcal{C}) \times_{\mathcal{C}} \text{Exc}(\mathcal{S}_{\text{fin},*}, \mathcal{C}),$$

and indeed there is such an equivalence under which an excisive functor $L : \mathcal{S}_{\text{fin},*}^2 \rightarrow \mathcal{C}$ corresponds to the pair of excisive functors

$$(L(*, -), L(-, *)),$$

and the pair (L_1, L_2) with $L_1(*) = L_2(*)$ corresponds to the excisive functor given by the fibre product

$$(X, Y) \mapsto L_1(X) \times_{L_1(*)=L_2(*)} L_2(Y).$$

That claim is proved in the next lemma.

LEMMA 8.4. *Let $\mathcal{S}_1, \mathcal{S}_2$ be ∞ -categories each with finite colimits and a terminal object $*$, and let \mathcal{C} be a differentiable ∞ -category. Then a functor*

$$L : \mathcal{S}_1 \times \mathcal{S}_2 \rightarrow \mathcal{C}$$

is excisive (as a functor of one variable) if and only if

- (1) *L is excisive in each variable individually; and*
- (2) *The morphisms $X_1 \rightarrow *$ and $X_2 \rightarrow *$ determine equivalences*

$$L(X_1, X_2) \simeq L(X_1, *) \times_{L(*,*)} L(*, X_2)$$

for all $X_1 \in \mathcal{S}_1, X_2 \in \mathcal{S}_2$.

PROOF. Key to this result is the fact that a pushout square in the ∞ -category $\mathcal{S}_1 \times \mathcal{S}_2$ is a square for which each component is a pushout in its respective \mathcal{S}_i .

Suppose that L is excisive. Applying that condition to a pushout square in $\mathcal{S}_1 \times \mathcal{S}_2$ of the form

$$\begin{array}{ccc} (X_0, Y) & \longrightarrow & (X_1, Y) \\ \downarrow & & \downarrow \\ (X_2, Y) & \longrightarrow & (X_{12}, Y) \end{array}$$

consisting of an arbitrary pushout in \mathcal{S}_1 and a fixed object in \mathcal{S}_2 , we deduce that L is excisive in its \mathcal{S}_1 variable. Similarly for its \mathcal{S}_2 variable.

Applying the condition that L is excisive to a pushout square in $\mathcal{S}_1 \times \mathcal{S}_2$ of the form

$$\begin{array}{ccc} (X_1, X_2) & \longrightarrow & (X_1, *) \\ \downarrow & & \downarrow \\ (*, X_2) & \longrightarrow & (*, *) \end{array}$$

we deduce condition (2).

Conversely, suppose that L satisfies conditions (1) and (2), and consider a diagram

$$\begin{array}{ccc} (X_0, Y_0) & \longrightarrow & (X_1, Y_1) \\ \downarrow & & \downarrow \\ (X_2, Y_2) & \longrightarrow & (X_{12}, Y_{12}) \end{array}$$

that is a pushout in each component.

Consider the following diagram in \mathcal{C} :

$$\begin{array}{ccccc} L(X_0, Y_0) & \longrightarrow & L(X_0, Y_1) & \longrightarrow & L(X_1, Y_1) \\ \downarrow & & \downarrow & & \downarrow \\ L(X_2, Y_0) & \longrightarrow & L(X_2, Y_1) & \longrightarrow & L(X_{12}, Y_1) \\ \downarrow & & \downarrow & & \downarrow \\ L(X_2, Y_2) & \longrightarrow & L(X_2, Y_{12}) & \longrightarrow & L(X_{12}, Y_{12}) \end{array}$$

The top-right and bottom-left squares are pullbacks because L is excisive in each variable individually, so it is sufficient to show that the top-left and bottom-right squares are also pullbacks, since then the whole square is a pullback by standard pasting properties of pullbacks (see [Lur09a, 4.4.2.1]).

For the top-left square (the bottom-right is similar), consider the diagram

$$\begin{array}{ccccc} L(X_0, Y_0) & \longrightarrow & L(X_0, Y_1) & \longrightarrow & L(X_0, *) \\ \downarrow & & \downarrow & & \downarrow \\ L(X_2, Y_0) & \longrightarrow & L(X_2, Y_1) & \longrightarrow & L(X_2, *) \\ \downarrow & & \downarrow & & \downarrow \\ L(*, Y_0) & \longrightarrow & L(*, Y_1) & \longrightarrow & L(*, *) \end{array}$$

Condition (2) implies that the bottom-right square, the bottom half, right-hand half, and entire square are all pullbacks. From that it follows by a succession of

applications of [Lur09a, 4.4.2.1] that each individual square is a pullback too, including the top-left as required. \square

A crucial role in the following sections will be played by a universal A -excisive approximation. The next proposition is a generalization of 7.6, which corresponds to the case $A = W$.

PROPOSITION 8.5. *Let A be a Weil-algebra with n generators, and let \mathcal{C} be a differentiable ∞ -category. Then there is a functor*

$$P_A : \text{Fun}(\mathcal{S}_{\text{fin},*}^n, \mathcal{C}) \rightarrow \text{Exc}^A(\mathcal{S}_{\text{fin},*}^n, \mathcal{C})$$

that is left adjoint to the inclusion and preserves finite limits. Moreover the ∞ -category $T^A(\mathcal{C}) = \text{Exc}^A(\mathcal{S}_{\text{fin},*}^n, \mathcal{C})$ is differentiable with finite limits and sequential colimits all calculated objectwise in \mathcal{C} . We write

$$p_A : F \rightarrow P_A(F)$$

for the corresponding universal P_A -approximation map.

PROOF. The first part is an example of the multivariable excisive approximation construction described in Definition 7.9. It follows from the definition of excisive, and the fact that \mathcal{C} is differentiable, that the subcategory $\text{Exc}^A(\mathcal{S}_{\text{fin},*}^n, \mathcal{C})$ of $\text{Fun}(\mathcal{S}_{\text{fin},*}^n, \mathcal{C})$ is closed under finite limits and sequential colimits which are computed objectwise in \mathcal{C} , and hence commute. Therefore $\text{Exc}^A(\mathcal{S}_{\text{fin},*}^n, \mathcal{C})$ is differentiable. \square

The following property of P_A is used in the proof of Lemma 9.14.

LEMMA 8.6. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between differentiable ∞ -categories that preserves both finite limits and sequential colimits. Then the following diagram commutes (up to natural equivalence)*

$$\begin{array}{ccc} \text{Fun}(\mathcal{S}_{\text{fin},*}^n, \mathcal{C}) & \xrightarrow{P_A} & \text{Exc}^A(\mathcal{S}_{\text{fin},*}^n, \mathcal{C}) \\ F_* \downarrow & & \downarrow F_* \\ \text{Fun}(\mathcal{S}_{\text{fin},*}^n, \mathcal{D}) & \xrightarrow{P_A} & \text{Exc}^A(\mathcal{S}_{\text{fin},*}^n, \mathcal{D}) \end{array}$$

where each functor F_* denotes post-composition with F .

PROOF. This claim is a multivariable version of [Lur17, 6.1.1.32] and follows from the explicit construction of the excisive approximation, as in the proof of Lemma 7.8. \square

Tangent structure on morphisms in $\text{Cat}_{\infty}^{\text{diff}}$. We now turn to the action of the tangent structure functors T^A on morphisms in $\text{Cat}_{\infty}^{\text{diff}}$, i.e. functors $F : \mathcal{C} \rightarrow \mathcal{D}$ between differentiable ∞ -categories that preserve sequential colimits. This action is the obvious extension of that described in 7.14 for the tangent bundle functor T .

DEFINITION 8.7. Let A be a Weil-algebra, and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between differentiable ∞ -categories that preserves sequential colimits. We define

$$T^A(F) : T^A(\mathcal{C}) \rightarrow T^A(\mathcal{D})$$

to be the composite

$$\text{Exc}^A(\mathcal{S}_{\text{fin},*}^n, \mathcal{C}) \subseteq \text{Fun}(\mathcal{S}_{\text{fin},*}^n, \mathcal{C}) \xrightarrow{F_*} \text{Fun}(\mathcal{S}_{\text{fin},*}^n, \mathcal{D}) \xrightarrow{P_A} \text{Exc}^A(\mathcal{S}_{\text{fin},*}^n, \mathcal{D}).$$

That is, we have

$$(8.8) \quad T^A(F)(L) := P_A(FL)$$

for an A -excisive functor $L : \mathcal{S}_{\text{fin},*}^n \rightarrow \mathcal{C}$.

LEMMA 8.9. *Let A be a Weil-algebra, and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a sequential-colimit-preserving functor between differential ∞ -categories. Then $T^A(F)$ also preserves sequential colimits.*

PROOF. Each of the functors F_* , P_A in Definition 8.7 preserves sequential colimits which, in all cases, are computed pointwise in the ∞ -categories \mathcal{C} and \mathcal{D} . \square

We now check that Definition 8.7 makes T^A into a functor $\text{Cat}_\infty^{\text{diff}} \rightarrow \text{Cat}_\infty^{\text{diff}}$, at least up to higher equivalence.

LEMMA 8.10. *Let A be a Weil-algebra. Then:*

- (1) *for the identity functor $I_{\mathcal{C}}$ on a differential ∞ -category \mathcal{C} , there is a natural equivalence*

$$I_{T^A(\mathcal{C})} \xrightarrow{\sim} T^A(I_{\mathcal{C}})$$

given by the maps $p_A : L \xrightarrow{\sim} P_A(L)$ for A -excisive $L : \mathcal{S}_{\text{fin},}^n \rightarrow \mathcal{C}$;*

- (2) *for sequential-colimit-preserving functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$ between differential ∞ -categories, there is a natural equivalence*

$$T^A(GF) \xrightarrow{\sim} T^A(G)T^A(F)$$

which comprises natural equivalences

$$P_A(GFL) \xrightarrow{\sim} P_A(GP_A(FL))$$

induced by the A -excisive approximation map $p_A : FL \rightarrow P_A(FL)$ for $L \in \text{Exc}^A(\mathcal{S}_{\text{fin},}^n, \mathcal{C})$.*

PROOF. Part (1) is a standard property of excisive approximation. Part (2) is more substantial. When $A = W$, so that $P_A = P_1$, this result is proved by Lurie in [Lur17, 7.3.1.14]. (That result is in the context of presentable ∞ -categories and functors that preserve all filtered colimits, but the proof works equally well for differentiable ∞ -categories and functors that only preserve sequential colimits. Fundamentally this result relies on the Klein-Rognes [KR02] Chain Rule as extended to ∞ -categories by Lurie in [Lur17, 6.2.1.24].)

In particular, Lurie's argument shows that when the functor $G : \mathcal{D} \rightarrow \mathcal{E}$ between differentiable ∞ -categories preserves sequential colimits, we have an equivalence

$$P_1(GH) \rightarrow P_1(GP_1(H))$$

for any $H : \mathcal{S}_{\text{fin},*} \rightarrow \mathcal{D}$. That argument extends to the case of $H : \mathcal{S}_{\text{fin},*}^n \rightarrow \mathcal{D}$, simply by replacing $\mathcal{S}_{\text{fin},*}$ with $\mathcal{S}_{\text{fin},*}^n$ and the null object $*$ with $(*, \dots, *)$. This observation provides the desired result when $A = W^n$.

Now consider an arbitrary Weil-algebra $A = W^{n_1} \otimes \dots \otimes W^{n_r}$. Then we have

$$P_A = P_1^{(r)} \dots P_1^{(1)}$$

where $P_1^{(j)}$ is the ordinary excisive approximation applied to a functor

$$\mathcal{S}_{\text{fin},*}^{n_1} \times \dots \times \mathcal{S}_{\text{fin},*}^{n_r} \rightarrow \mathcal{D}$$

in its $\mathcal{S}_{\text{fin},*}^{n_j}$ variable. For any $H : \mathcal{S}_{\text{fin},*}^n \rightarrow \mathcal{D}$, the map $p_A : H \rightarrow P_A(H)$ factors as a composite

$$H \xrightarrow{p_1^{(1)}} P_1^{(1)} H \xrightarrow{p_1^{(2)}} P_1^{(2)} P_1^{(1)} H \xrightarrow{p_1^{(3)}} \dots \xrightarrow{p_1^{(r)}} P_A H$$

of excisive approximations in each variable separately. Applying G to the j -th map yields a $P_1^{(j)}$ -equivalence by the extended version of Lurie's argument, and hence a P_A -equivalence as required. Thus we have an equivalence

$$(8.11) \quad P_A(GH) \xrightarrow{\sim} P_A(GP_A(H)).$$

Taking H to be FL yields the desired result. \square

Tangent structure on morphisms in Weil. We next address functoriality of our putative tangent structure in the Weil variable. Let $\phi : A \rightarrow A'$ be a morphism in Weil, i.e. a labelled morphism of Weil-algebras, and let \mathcal{C} be a differentiable ∞ -category. We construct a (sequential-colimit-preserving) functor

$$T^\phi(\mathcal{C}) : T^A(\mathcal{C}) \rightarrow T^{A'}(\mathcal{C})$$

as follows. Roughly speaking, T^ϕ is the map

$$\text{Exc}^A(\mathcal{S}_{\text{fin},*}^n, \mathcal{C}) \rightarrow \text{Exc}^{A'}(\mathcal{S}_{\text{fin},*}^{n'}, \mathcal{C})$$

given by precomposition with a suitable functor

$$\tilde{\phi} : \mathcal{S}_{\text{fin},*}^{n'} \rightarrow \mathcal{S}_{\text{fin},*}^n$$

whose definition mirrors the Weil-algebra morphism ϕ .

DEFINITION 8.12. Let $\phi : A \rightarrow A'$ be a labelled morphism between Weil-algebras with n and n' generators respectively. Recall from Definition 2.4 that ϕ is a span of finite partial commutative monoids of the form

$$\begin{array}{ccc} & K & \\ s \swarrow & & \searrow t \\ M_A & & M_{A'} \end{array}$$

where

$$\phi(x_i) = \sum_{k \in s^{-1}(x_i)} t(k)$$

for each generator x_i of the Weil-algebra A . We define a corresponding functor

$$\tilde{\phi} : \mathcal{S}_{\text{fin},*}^{n'} \rightarrow \mathcal{S}_{\text{fin},*}^n$$

by setting

$$\tilde{\phi}(Y_1, \dots, Y_{n'}) := (X_1, \dots, X_n)$$

where

$$X_i := \bigvee_{k \in s^{-1}(x_i)} Y_{t(k)}$$

where a monomial $t(k) = y_{j_1} \cdots y_{j_r}$ in A' determines a corresponding smash product

$$Y_{t(k)} := Y_{j_1} \wedge \dots \wedge Y_{j_r}.$$

Note that if $s^{-1}(x_i)$ is empty, i.e. if $\phi(x_i) = 0$, then we set $X_i := *$.

DEFINITION 8.13. Let $\phi : A \rightarrow A'$ be a labelled Weil-algebra morphism, and let \mathcal{C} be a differential ∞ -category. Then we define

$$T^\phi(\mathcal{C}) : T^A(\mathcal{C}) \rightarrow T^{A'}(\mathcal{C}); \quad L \mapsto P_{A'}(L\tilde{\phi}),$$

that is, $T^\phi(\mathcal{C})$ is the composite

$$\mathrm{Exc}^A(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C}) \xrightarrow{\tilde{\phi}^*} \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n'}, \mathcal{C}) \xrightarrow{P_{A'}} \mathrm{Exc}^{A'}(\mathcal{S}_{\mathrm{fin},*}^{n'}, \mathcal{C})$$

where $\tilde{\phi}^*$ denotes precomposition with $\tilde{\phi}$.

EXAMPLES 8.14. Using Definition 8.13, we can now work out how the fundamental natural transformations from Cockett and Cruttwell's definition of tangent structure [CC14, 2.3] manifest in our case.

- (1) Let $\epsilon : W \rightarrow \mathbb{N}$ be the augmentation. Then $\tilde{\epsilon} : * \rightarrow \mathcal{S}_{\mathrm{fin},*}$ is the functor that picks out the null object $*$, and so the *projection*

$$p := T^\epsilon : T(\mathcal{C}) \rightarrow \mathcal{C}$$

can be identified with the evaluation map

$$\mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}) \rightarrow \mathcal{C}; \quad L \mapsto L(*),$$

as in Definition 7.12.

- (2) Let $\eta : \mathbb{N} \rightarrow W$ be the unit map. Then $\tilde{\eta} : \mathcal{S}_{\mathrm{fin},*} \rightarrow *$ is, of course, the constant map, and so the *zero section*

$$0 := T^\eta : \mathcal{C} \rightarrow T(\mathcal{C})$$

can be identified with the map

$$\mathcal{C} \rightarrow \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}); \quad C \mapsto \mathrm{const}_C$$

that picks out the constant functors.

- (3) Let $\phi : W^2 \rightarrow W$ be the addition map given by $x \mapsto x$ and $y \mapsto x$. Then $\tilde{\phi} : \mathcal{S}_{\mathrm{fin},*} \rightarrow \mathcal{S}_{\mathrm{fin},*}^2$ is the diagonal $X \mapsto (X, X)$, and so the *addition*

$$+ := T^\phi : T^{W^2}(\mathcal{C}) \rightarrow T(\mathcal{C})$$

is given by the map

$$\mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}^2, \mathcal{C}) \rightarrow \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}); \quad L \mapsto (X \mapsto L(X, X)).$$

Under the equivalence $T^{W^2}(\mathcal{C}) \simeq T(\mathcal{C}) \times_{\mathcal{C}} T(\mathcal{C})$ described in 8.3(4), we can identify $+$ with the fibrewise product map

$$\begin{aligned} \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}) \times_{\mathcal{C}} \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}) &\rightarrow \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}); \\ (L_1, L_2) &\mapsto L_1(-) \times_{L_1(*)=L_2(*)} L_2(-). \end{aligned}$$

- (4) Let $\sigma : W \otimes W \rightarrow W \otimes W$ be the symmetry map. Then $\tilde{\sigma} : \mathcal{S}_{\mathrm{fin},*}^2 \rightarrow \mathcal{S}_{\mathrm{fin},*}^2$ is given by $(X, Y) \mapsto (Y, X)$, and the *flip*

$$c := T^\sigma : T^2(\mathcal{C}) \rightarrow T^2(\mathcal{C})$$

is the symmetry map

$$\mathrm{Exc}^{1,1}(\mathcal{S}_{\mathrm{fin},*}^2, \mathcal{C}) \rightarrow \mathrm{Exc}^{1,1}(\mathcal{S}_{\mathrm{fin},*}^2, \mathcal{C}); \quad L \mapsto [(X, Y) \mapsto L(Y, X)].$$

satisfying the conditions of Definition 2.17, and suppose the Weil-algebras A, A', A'' have n, n', n'' generators respectively.

The functor $\widetilde{\phi_2\phi_1}$ is given by

$$\widetilde{\phi_2\phi_1}(Z_1, \dots, Z_{n''}) = (X_1, \dots, X_n)$$

where

$$X_i = \bigvee_{k'' : s''(k'')=x_i} Z_{t''(k'')}.$$

On the other hand, the composite functor $\widetilde{\phi_1\phi_2}$ is given by

$$\widetilde{\phi_1\phi_2}(Z_1, \dots, Z_n) \cong (X'_1, \dots, X'_n)$$

where

$$X'_i = \bigvee_{k : s(k)=x_i} \left(\bigvee_{k'_1, \dots, k'_r \in K'_1 : t(k)=s'(k'_1)\cdots s'(k'_r)} Z_{t'(k_1)} \wedge \dots \wedge Z_{t'(k_r)} \right).$$

We define the natural transformation

$$\alpha : \widetilde{\phi_2\phi_1} \rightarrow \widetilde{\phi_1\phi_2}$$

by the maps $X_i \rightarrow X'_i$ which on the factor corresponding to $k'' \in K''$ such that $s''(k'') = x_i$, is the canonical isomorphism

$$\alpha_{k''} : Z_{t''(k'')} \rightarrow Z_{t'(k'_1)} \wedge \dots \wedge Z_{t'(k'_r)}$$

into the factor given by $k = s'''(k'')$ and $k'_1 \cdots k'_r = t'''(k'')$, noting that

$$t''(k'') = t'(t'''(k)) = t'(k'_1 \cdots k'_r) = t'(k'_1) \cdots t'(k'_r).$$

Note that no monomial in a Weil-algebra has a repeated factor, so there is indeed only one canonical isomorphism $\alpha_{k''}$ which matches up the factors in the two smash products.

REMARK 8.19. The pullback condition in the definition of a 2-simplex in $\mathbb{W}\text{eil}$ identifies the set K''_i with pairs $(k, k'_1 \cdots k'_r)$ such that $s(k) = x_i$, and $t(k) = s'(k'_1) \cdots s'(k'_r)$, and such that the product $k'_1 \cdots k'_r$ is defined in K' . However, there are other terms appearing in the wedge sum that forms X'_i : those where $k'_1 \cdots k'_r$ is not an element of K' because $t'(k'_1) \cdots t'(k'_r)$ is zero in A'' .

It follows that α is the inclusion of the wedge summand in a decomposition

$$(8.20) \quad \widetilde{\phi_1\phi_2} \simeq \widetilde{\phi_2\phi_1} \vee \zeta(\phi_1, \phi_2)$$

where $\zeta(\phi_1, \phi_2)(Z_1, \dots, Z_{n''})$ is a finite wedge sum of terms, each of the form

$$Z_{k_1} \wedge \dots \wedge Z_{k_q}$$

where the monomial $z_{k_1} \cdots z_{k_q}$ is zero in the Weil-algebra A'' .

PROOF OF LEMMA 8.17. The calculation of $\tilde{\Gamma}_A$ in part (1) follows immediately from Definition 8.12. Part (2) is much more substantial and will occupy the next several pages. We prove that each of the two maps (i) and (ii) is an equivalence, and both of those facts are important later in the paper.

For (i), let $L : \mathcal{S}_{\text{fin},*}^n \rightarrow \mathcal{C}$ be an A -excisive functor. We have to show that the map

$$(8.21) \quad P_{A''}(L\alpha) : P_{A''}(L\widetilde{\phi_1\phi_2}) \rightarrow P_{A''}(L\widetilde{\phi_2\phi_1})$$

is an equivalence. The idea here is that the difference between $\widetilde{\phi_1\phi_2}$ and $\widetilde{\phi_2\phi_1}$, i.e. the factor $\zeta(\phi_1, \phi_2)$ in (8.20), consists of terms that make no contribution after taking the A'' -excisive approximation.

By induction on the number of wedge summands in $\zeta(\phi_1, \phi_2)$, we reduce to showing that for any functor $\beta : \mathcal{S}_{\text{fin},*}^{n''} \rightarrow \mathcal{S}_{\text{fin},*}^n$, the inclusion $\beta \rightarrow \beta \vee \xi$ induces an equivalence

$$P_{A''}(L\beta) \rightarrow P_{A''}(L(\beta \vee \xi))$$

where

$$\xi(Z_1, \dots, Z_{n''}) \simeq (*, \dots, *, Z_{k_1} \wedge \dots \wedge Z_{k_q}, *, \dots, *)$$

for some sequence of indices k_1, \dots, k_r such that $z_{k_i} z_{k_{i'}} = 0$ in A'' for some $i \neq i'$. Equivalently, we show that the collapse map $\beta \vee \xi \rightarrow \beta$ induces an equivalence in the other direction.

Writing $L : \mathcal{S}_{\text{fin},*}^{n_1} \times \dots \times \mathcal{S}_{\text{fin},*}^{n_r} \rightarrow \mathcal{C}$ according to the decomposition of the Weil algebra $A = W^{n_1} \otimes \dots \otimes W^{n_r}$, we can assume without loss of generality that $r = 1$ (i.e. $A = W^n$) since ξ is only non-trivial in one variable. In that case $L : \mathcal{S}_{\text{fin},*}^n \rightarrow \mathcal{C}$ is excisive. Applying L to the pushout diagram in $\text{Fun}(\mathcal{S}_{\text{fin},*}^{n''}, \mathcal{S}_{\text{fin},*}^n)$ of the form

$$\begin{array}{ccc} \beta \vee \xi & \longrightarrow & \beta \\ \downarrow & & \downarrow \\ \xi & \longrightarrow & * \end{array}$$

and taking $P_{A''}$ (which preserves pullbacks), we get a pullback square

$$\begin{array}{ccc} P_{A''}(L(\beta \vee \xi)) & \longrightarrow & P_{A''}(L\beta) \\ \downarrow & & \downarrow \\ P_{A''}(L\xi) & \longrightarrow & P_{A''}(L(*)) \end{array}$$

so it is sufficient to show that the bottom map is an equivalence. Replacing \mathcal{C} with the slice ∞ -category $\mathcal{C}_{/L(*)}$, we can assume that L is reduced. Our goal is then to show that the functor

$$(8.22) \quad (Z_1, \dots, Z_{n''}) \mapsto L(*, \dots, *, Z_{k_1} \wedge \dots \wedge Z_{k_q}, *, \dots, *)$$

has trivial A'' -excisive part whenever $z_{k_i} z_{k_{i'}} = 0$ in A'' for some $i \neq i'$.

If we write

$$\mathcal{S}_{\text{fin},*}^{n''} = \mathcal{S}_{\text{fin},*}^{n_1''} \times \dots \times \mathcal{S}_{\text{fin},*}^{n_s''}$$

according to the decomposition $A'' = W^{n_1''} \otimes \dots \otimes W^{n_s''}$, then the condition that $z_{k_i} z_{k_{i'}} = 0$ implies that the variables Z_{k_i} and $Z_{k_{i'}}$ are in the same factor $\mathcal{S}_{\text{fin},*}^{n_j''}$, and it is sufficient to show that the functor (8.22) has trivial excisive approximation with respect to that factor. Without loss of generality, we can take $A'' = W^{n''}$ so that we simply have to show that a functor $F : \mathcal{S}_{\text{fin},*}^{n''} \rightarrow \mathcal{C}$ of the form (8.22) has trivial excisive approximation whenever $q \geq 2$.

Since F is reduced in each of the variables Z_{k_i} , it is, by [Lur17, 6.1.3.10], q -reduced when viewed as a functor of all of those variables. Since $q \geq 2$, it follows that F has trivial excisive approximation with respect to those variables. It follows

from Lemma 8.4 that the excisive approximation of a functor $\mathcal{S}_{\text{fin},*}^{n''} \rightarrow \mathcal{C}$ factors via its excisive approximation with respect to any subset of its variables, so F also has trivial excisive approximation as a functor $\mathcal{S}_{\text{fin},*}^{n''} \rightarrow \mathcal{C}$, as desired. This completes the proof that the map (i) is an equivalence.

For (ii), we prove the following: for any $G : \mathcal{S}_{\text{fin},*}^{n'_1} \times \cdots \times \mathcal{S}_{\text{fin},*}^{n'_r} \rightarrow \mathcal{C}$ and any Weil-algebra morphism $\phi : A' \rightarrow A''$, there is an equivalence

$$(8.23) \quad P_{A''}(G\tilde{\phi}) \xrightarrow{\sim} P_{A''}(P_{A'}(G)\tilde{\phi})$$

induced by the A' -excisive approximation map $p_{A'} : G \rightarrow P_{A'}G$.

It is sufficient to show that excisive approximation with respect to each of the r variables induces an equivalence. Thus we can reduce to the case that $A' = W^{n'}$, in which case $P_{A'} = P_1$. Replacing \mathcal{C} with the slice ∞ -category $\mathcal{C}_{G(*)}$ of objects over and under $G(*)$, we can also assume that \mathcal{C} is a pointed ∞ -category and that G is reduced, i.e. $G(*) \simeq *$. So we have to show that for any reduced $G : \mathcal{S}_{\text{fin},*}^{n'} \rightarrow \mathcal{C}$ the map

$$(8.24) \quad P_{A''}(G\tilde{\phi}) \rightarrow P_{A''}((P_1G)\tilde{\phi})$$

is an equivalence.

We break this proof into two parts. First we use downward induction on the Taylor tower of G to reduce to the case G is m -homogeneous for some $m \geq 2$. From there we use the specific form of $\tilde{\phi}$, and the fact that ϕ is an algebra homomorphism, to show (8.24) is an equivalence by direct calculation.

To start the induction we apply Lemma 7.8 which tells us that, since $\tilde{\phi}$ is reduced, the universal n -excisive approximation $p_n : G \rightarrow P_nG$ induces an equivalence

$$(8.25) \quad P_n(G\tilde{\phi}) \xrightarrow{\sim} P_n((P_nG)\tilde{\phi})$$

for any n .

If $n \geq s$, then an A'' -excisive functor $\mathcal{S}_{\text{fin},*}^{n''_1} \times \cdots \times \mathcal{S}_{\text{fin},*}^{n''_s} \rightarrow \mathcal{C}$ is n -excisive by [Lur17, 6.1.3.4]. It follows from (8.25) that p_n induces an equivalence

$$(8.26) \quad P_{A''}(G\tilde{\phi}) \xrightarrow{\sim} P_{A''}((P_nG)\tilde{\phi}).$$

Next consider the fibre sequences

$$P_mG \rightarrow P_{m-1}G \rightarrow R_mG$$

provided by [Lur17, 6.1.2.4] (Goodwillie's delooping theorem for homogeneous functors) in which R_mG is m -homogeneous. We show below that

$$(8.27) \quad P_{A''}(H\tilde{\phi}) \simeq *$$

for any functor $H : \mathcal{S}_{\text{fin},*}^{n'} \rightarrow \mathcal{C}$ that is m -homogeneous for some $m \geq 2$. Since $P_{A''}$ preserves fibre sequences, we then deduce that for $m \geq 2$

$$P_{A''}((P_mG)\tilde{\phi}) \xrightarrow{\sim} P_{A''}((P_{m-1}G)\tilde{\phi}),$$

which combined with (8.26) implies the desired result (8.24).

So our goal now is (8.27). By [Lur17, 6.1.2.9], we can assume that \mathcal{C} is a stable ∞ -category, and then Lemma 8.31 below gives a classification of m -homogeneous functors $\mathcal{S}_{\text{fin},*}^{n'} \rightarrow \mathcal{C}$. That result tells us that H is a finite product of terms of the form

$$(8.28) \quad (Y_1, \dots, Y_{n'}) \mapsto L(Y_1^{\wedge m_1} \wedge \cdots \wedge Y_{n'}^{\wedge m_{n'}})_{h(\Sigma_{m_1} \times \cdots \times \Sigma_{m_{n'}})}$$

for linear $L : \mathcal{S}_{\text{fin},*} \rightarrow \mathcal{C}$ and $m = m_1 + \cdots + m_{n'}$. Since $P_{A''}$ preserves finite products, we can reduce our goal (8.27) to the case that H is of the form in (8.28).

The next part of the proof depends essentially on the fact that $\phi : A' \rightarrow A''$ is an algebra homomorphism, and we start by describing what that condition implies about the functor $\tilde{\phi}$.

Recall that we have reduced to the case

$$A' = W^{n'} = \mathbb{N}[y_1, \dots, y_{n'}] / (y_i y_{i'})_{i, i'=1, \dots, n'}$$

and let us denote the generators of

$$A'' = W^{n''_1} \otimes \cdots \otimes W^{n''_s}$$

as $z_{j,1}, \dots, z_{j,n''_j}$ for $j = 1, \dots, s$. Similarly, we denote inputs to the corresponding functor

$$\tilde{\phi} : \mathcal{S}_{\text{fin},*}^{n''_1} \times \cdots \times \mathcal{S}_{\text{fin},*}^{n''_s} \rightarrow \mathcal{S}_{\text{fin},*}^{n'}$$

with the notation $Z_{j,k}$. We write \underline{Z} for the object of $\mathcal{S}_{\text{fin},*}^{n''_1} \times \cdots \times \mathcal{S}_{\text{fin},*}^{n''_s}$ with these entries.

Since ϕ is an algebra homomorphism, we have, for any i, i' :

$$\phi(y_i)\phi(y_{i'}) = \phi(y_i y_{i'}) = \phi(0) = 0.$$

There are no terms in A'' with negative coefficients, so this means that the product of any monomial in $\phi(y_i)$ and any monomial in $\phi(y_{i'})$ must contain a factor of the form $z_{j,k} z_{j,k'}$ for some j, k, k' .

Translating this observation into a statement about the map $\tilde{\phi}$, and writing $\tilde{\phi}_1, \dots, \tilde{\phi}_{n'}$ for the components of that map, we see that for all $i, i' = 1, \dots, n'$:

$$(8.29) \quad \tilde{\phi}_i(\underline{Z}) \wedge \tilde{\phi}_{i'}(\underline{Z}) \simeq \bigvee Z_{j_1, k_1} \wedge \cdots \wedge Z_{j_t, k_t},$$

a finite wedge sum of terms each of which satisfies $j_l = j_{l'}$ for some $l \neq l'$.

Now consider the functor $H\tilde{\phi} : \mathcal{S}_{\text{fin},*}^{n''} \rightarrow \mathcal{C}$ where H is of the form (8.28) with $m = m_1 + \cdots + m_{n'} \geq 2$. Applying H to $\tilde{\phi}$ that satisfies (8.29), and since the linear functor L takes finite wedge sums to products, we obtain a decomposition of the form

$$(8.30) \quad H\tilde{\phi}(\underline{Z}) \simeq \prod L(Z_{j_1, k_1} \wedge \cdots \wedge Z_{j_t, k_t})_{hG}$$

a product of terms where $L : \mathcal{S}_{\text{fin},*} \rightarrow \mathcal{C}$ is linear, $j_l = j_{l'}$ for some $l \neq l'$, and G is a subgroup of Σ_m that acts by permuting some of the factors $Z_{j,k}$.

By Lemma 8.31 below, each of the terms in (8.30) is r_j -homogeneous in the variable $\mathcal{S}_{\text{fin},*}^{n''_j}$ where r_j is the number of times that the index j appears in the list j_1, \dots, j_t . We know that there is some j such that $r_j \geq 2$, which implies that each of these terms has trivial A'' -excisive approximation. We therefore obtain the desired (8.27):

$$P_{A''}(H\tilde{\phi}) \simeq *.$$

Putting this calculation together with our earlier induction, we get the equivalence (8.23) which completes the proof that map (ii) is an equivalence, once we have proved the following lemma. \square

LEMMA 8.31. *Let \mathcal{C} be a stable ∞ -category. A functor $H : \mathcal{S}_{\text{fin},*}^k \rightarrow \mathcal{C}$ is m -homogeneous if and only if it can be written in the form*

$$H(Y_1, \dots, Y_k) \simeq \prod_{m_1 + \dots + m_k = m} L_{(m_1, \dots, m_k)}(Y_1^{\wedge m_1} \wedge \dots \wedge Y_k^{\wedge m_k})_{h(\Sigma_{m_1} \times \dots \times \Sigma_{m_k})}$$

for a collection of linear (i.e. reduced and excisive) functors

$$L_{(m_1, \dots, m_k)} : \mathcal{S}_{\text{fin},*} \rightarrow \mathcal{C}$$

indexed by the ordered partitions of m into k non-negative integers. The right-hand side of this equivalence involves the (homotopy) coinvariants for the action of the group $\Sigma_{m_1} \times \dots \times \Sigma_{m_k}$ that permutes each of the smash powers $Y_i^{\wedge m_i}$.

PROOF. Suppose H is m -homogeneous, and let $\underline{Y} = (Y_1, \dots, Y_k)$ be an object of $\mathcal{S}_{\text{fin},*}^k$. By [Lur17, 6.1.4.14], we can write

$$H(\underline{Y}) \simeq M(\underline{Y}, \dots, \underline{Y})_{h\Sigma_m}$$

where $M : (\mathcal{S}_{\text{fin},*}^k)^m \rightarrow \mathcal{C}$ is symmetric multilinear. By induction on Lemma 8.4, a linear functor $\mathcal{S}_{\text{fin},*}^k \rightarrow \mathcal{C}$ is of the form

$$L_1(Y_1) \times \dots \times L_k(Y_k)$$

and it follows that the symmetric multilinear M can be written as

$$M(\underline{Y}_1, \dots, \underline{Y}_m) \simeq \prod_{1 \leq j_1, \dots, j_m \leq k} L_{j_1, \dots, j_m}(Y_{1, j_1}, \dots, Y_{m, j_m})$$

where the symmetric group Σ_m acts by permuting the indexes j_1, \dots, j_m as well as the inputs of each multilinear functor $L_{j_1, \dots, j_m} : \mathcal{S}_{\text{fin},*}^m \rightarrow \mathcal{C}$.

We therefore get

$$H(\underline{Y}) \simeq \prod_{m_1 + \dots + m_k = m} L_{1, \dots, 1, \dots, k, \dots, k}(Y_1, \dots, Y_1, \dots, Y_k, \dots, Y_k)_{h(\Sigma_{m_1} \times \dots \times \Sigma_{m_k})}$$

where the index i is repeated m_i times in each term. Finally, by induction using the result of [Lur17, 1.4.2.22], multilinear functors $\mathcal{S}_{\text{fin},*}^m \rightarrow \mathcal{C}$ factor via the smash product $\wedge : \mathcal{S}_{\text{fin},*}^m \rightarrow \mathcal{S}_{\text{fin},*}$ yielding the desired expression.

Conversely, suppose H is of the given form. It is sufficient to show that each functor of the form

$$F(Y_1, \dots, Y_k) \mapsto L(Y_1^{\wedge m_1} \wedge \dots \wedge Y_k^{\wedge m_k})$$

with L linear, is m -homogeneous, since the finite product and coinvariants constructions preserve homogeneity of functors with values in a stable ∞ -category. It can be shown directly from the definition that F is m_i -excisive in its i -th variable, so that F is m -excisive by [Lur17, 6.1.3.4]. To see that F is also m -reduced, we apply [Lur17, 6.1.3.24] by directly calculating the relevant cross-effects of F . \square

Finally for this section, we note the following compatibility between natural transformations of the form α which will be crucial for establishing the higher homotopy coherence of our constructions in Chapter 9.

LEMMA 8.32. *Let σ be a 3-simplex in Weil with edges given by the labelled Weil-algebra morphisms $\phi_1, \phi_2, \phi_3, \phi_{12}, \phi_{23}, \phi_{123}$. Then the natural transformations*

determined by applying Definition 8.18 to the 2-faces of σ form a strictly commutative diagram of functors $\mathcal{S}_{\text{fin},*}^{n_3} \rightarrow \mathcal{S}_{\text{fin},*}^{n_0}$ of the form:

$$\begin{array}{ccc} \widetilde{\phi_3 \phi_2 \phi_1} & \xrightarrow{\alpha_{12,3}} & \widetilde{\phi_2 \phi_1 \phi_3} \\ \downarrow \alpha_{1,23} & & \downarrow \alpha_{1,2\tilde{\phi}_3} \\ \widetilde{\tilde{\phi}_1 \phi_3 \phi_2} & \xrightarrow{\tilde{\phi}_1 \alpha_{2,3}} & \widetilde{\tilde{\phi}_1 \tilde{\phi}_2 \tilde{\phi}_3} \end{array}$$

Similarly, any m -simplex in Weil , for $m \geq 3$, determines a strictly commutative $(m-1)$ -cube of functors $\mathcal{S}_{\text{fin},*}^{n_m} \rightarrow \mathcal{S}_{\text{fin},*}^{n_0}$.

PROOF. The components of $\widetilde{\phi_3 \phi_2 \phi_1}$ are wedge summands indexed by elements $k \in \tau([0, 3])$. The pullback condition in Definition 2.17 implies that each such k is uniquely determined by elements $k_1 \in \tau([0, 1]), k_2 \in \tau([1, 2]), k_3 \in \tau([2, 3])$ which satisfy $t(k_1) = s'(k_2)$ and $t'(k_2) = s''(k_3)$. Following through Definition 8.18, we see that each composite map in the square above maps the wedge summand corresponding to k by the canonical isomorphism to the wedge summand of $\widetilde{\tilde{\phi}_1 \tilde{\phi}_2 \tilde{\phi}_3}$ corresponding to the triple (k_1, k_2, k_3) . Therefore, the diagram strictly commutes.

For an m -simplex, τ , the same argument implies that each 2-dimensional face of the corresponding $(m-1)$ -cube commutes, hence so does the whole cube. \square

Basic tangent structure properties. We show in Chapter 9 that the constructions of Definitions 8.1, 8.7 and 8.13 extend to a functor, i.e. map of simplicial sets,

$$T : \text{Weil} \times \text{Cat}_{\infty}^{\text{diff}} \rightarrow \text{Cat}_{\infty}^{\text{diff}}.$$

But first we check that our definitions so far satisfy the various conditions needed to determine a tangent structure on $\text{Cat}_{\infty}^{\text{diff}}$, starting with the observation that T corresponds to an action of the strict monoidal ∞ -category Weil on $\text{Cat}_{\infty}^{\text{diff}}$.

LEMMA 8.33. *Let \mathcal{C} be a differentiable ∞ -category. Then:*

(1) *there is a natural isomorphism $T^{\mathbb{N}}(\mathcal{C}) \cong \mathcal{C}$ given by evaluation at the unique point in $\mathcal{S}_{\text{fin},*}^0 = *$;*

(2) *for Weil-algebras A, A' , there is a natural isomorphism*

$$T^{A'}(T^A(\mathcal{C})) \cong T^{A \otimes A'}(\mathcal{C})$$

given by restricting the isomorphism

$$\text{Fun}(\mathcal{S}_{\text{fin},*}^{n'}, \text{Fun}(\mathcal{S}_{\text{fin},*}^n, \mathcal{C})) \cong \text{Fun}(\mathcal{S}_{\text{fin},*}^{n+n'}, \mathcal{C})$$

to the subcategories of suitably excisive functors.

PROOF. These results follow immediately from the definition of $T^A(\mathcal{C})$ in (8.2). \square

We now verify that the tangent pullbacks in Weil (see Proposition 2.31) are preserved by the action map T .

LEMMA 8.34. *Let J, J' be disjoint finite sets, and \mathcal{C} a differentiable ∞ -category. Then there is a pullback of ∞ -categories of the form*

$$\begin{array}{ccc} T_{J \sqcup J'}(\mathcal{C}) & \longrightarrow & T_J(\mathcal{C}) \\ \downarrow & & \downarrow \\ T_{J'}(\mathcal{C}) & \longrightarrow & \mathcal{C} \end{array}$$

where the horizontal maps are induced by the projection $T_{J'} \rightarrow 1$, and the vertical by $T_J \rightarrow 1$.

PROOF. To prove this lemma, we should first specify precisely what we mean by a ‘pullback of ∞ -categories’. For the purposes of this proof, we take that condition to mean that the given diagram is a homotopy pullback in the Joyal model structure on simplicial sets, and hence a pullback in the ∞ -category $\mathbb{C}at_\infty$. (In the proof of Theorem 9.22 in the next chapter we show that this diagram is also a pullback in the subcategory $\mathbb{C}at_\infty^{\text{diff}}$.)

We can write the desired diagram in the following form

$$\begin{array}{ccc} \text{Exc}(\mathcal{S}_{\text{fin},*}^{J \sqcup J'}, \mathcal{C}) & \xrightarrow{p'_1} & \text{Exc}(\mathcal{S}_{\text{fin},*}^J, \mathcal{C}) \\ p_1 \downarrow & & \downarrow p \\ \text{Exc}(\mathcal{S}_{\text{fin},*}^{J'}, \mathcal{C}) & \xrightarrow{p'} & \mathcal{C} \end{array}$$

where each map is given by evaluating either the J -indexed variables or the J' -indexed variables at $*$.

First we argue that the map p is a fibration in the Joyal model structure. This is true for the corresponding projection

$$\text{Fun}(\mathcal{S}_{\text{fin},*}^J, \mathcal{C}) \rightarrow \mathcal{C}$$

because $* \rightarrow \mathcal{S}_{\text{fin},*}^J$ is a cofibration between cofibrant objects, and the Joyal model structure on simplicial sets is closed monoidal. Since $\text{Exc}(\mathcal{S}_{\text{fin},*}^J, \mathcal{C})$ is a full subcategory of $\text{Fun}(\mathcal{S}_{\text{fin},*}^J, \mathcal{C})$ that is closed under equivalences, p is also a fibration by [Lur09a, 2.4.6.5].

It is now sufficient to show that the induced map

$$\pi : \text{Exc}(\mathcal{S}_{\text{fin},*}^J \times \mathcal{S}_{\text{fin},*}^{J'}, \mathcal{C}) \rightarrow \text{Exc}(\mathcal{S}_{\text{fin},*}^J, \mathcal{C}) \times_{\mathcal{C}} \text{Exc}(\mathcal{S}_{\text{fin},*}^{J'}, \mathcal{C})$$

is an equivalence of ∞ -categories.

To see this, we define an inverse map

$$\iota : \text{Exc}(\mathcal{S}_{\text{fin},*}^J, \mathcal{C}) \times_{\mathcal{C}} \text{Exc}(\mathcal{S}_{\text{fin},*}^{J'}, \mathcal{C}) \rightarrow \text{Exc}(\mathcal{S}_{\text{fin},*}^J \times \mathcal{S}_{\text{fin},*}^{J'}, \mathcal{C})$$

that sends a pair (F_1, F_2) of excisive functors with the property that $F_1(*) = F_2(*)$, to the functor

$$(X_1, X_2) \mapsto F_1(X_1) \times_{F_2(*)} F_2(X_2).$$

It is simple to check that $\pi \iota \simeq \text{id}$, and it follows from Lemma 8.4 that $\iota \pi \simeq \text{id}$. \square

The final thing we check in this section is that $\mathcal{C}\text{at}_\infty^{\text{diff}}$ has finite products given by the ordinary product of ∞ -categories, which are preserved by the tangent bundle functor $T = T^W : \mathcal{C}\text{at}_\infty^{\text{diff}} \rightarrow \mathcal{C}\text{at}_\infty^{\text{diff}}$. This fact implies that the tangent structure on $\mathcal{C}\text{at}_\infty^{\text{diff}}$ is cartesian in the sense of Definition 5.1.

LEMMA 8.35. *For differentiable ∞ -categories $\mathcal{C}, \mathcal{C}'$, the product $\mathcal{C} \times \mathcal{C}'$ is differentiable, and is the product of \mathcal{C} and \mathcal{C}' in $\mathcal{C}\text{at}_\infty^{\text{diff}}$. Moreover, the projections determine an equivalence of ∞ -categories*

$$T(\mathcal{C} \times \mathcal{C}') \simeq T(\mathcal{C}) \times T(\mathcal{C}').$$

PROOF. Since limits and colimits in the product ∞ -category are detected in each term, the product $\mathcal{C} \times \mathcal{C}'$ is differentiable. A functor $\mathcal{D} \rightarrow \mathcal{C} \times \mathcal{C}'$, with \mathcal{D} differentiable, preserves sequential colimits if and only if each component preserves sequential colimits. It follows that $\mathcal{C} \times \mathcal{C}'$ is the product in $\mathcal{C}\text{at}_\infty^{\text{diff}}$. Finally, the isomorphism

$$\text{Fun}(\mathcal{S}_{\text{fin},*}, \mathcal{C}_1 \times \mathcal{C}_2) \cong \text{Fun}(\mathcal{S}_{\text{fin},*}, \mathcal{C}_1) \times \text{Fun}(\mathcal{S}_{\text{fin},*}, \mathcal{C}_2)$$

restricts to the ∞ -categories of excisive functors since a square in $\mathcal{C}_1 \times \mathcal{C}_2$ is a pullback if and only if it is a pullback in each factor. \square

The vertical lift axiom. The last substantial condition we need in order that the constructions earlier in this chapter underlie a tangent structure T on the ∞ -category $\mathcal{C}\text{at}_\infty^{\text{diff}}$ is that T preserves the vertical lift pullback (1.8) in the category Weil. Applying Definition 8.13 to the Weil-algebra morphisms that appear in that pullback square, and using Lemma 8.4, we reduce to the following result.

PROPOSITION 8.36. *Let \mathcal{C} be a differentiable ∞ -category. Then the following diagram is a pullback of ∞ -categories:*

$$\begin{array}{ccc} T(\mathcal{C}) \times_{\mathcal{C}} T(\mathcal{C}) & \xrightarrow{v} & T(T(\mathcal{C})) \\ \downarrow p \times p & & \downarrow T(p) \\ \mathcal{C} & \xrightarrow{0} & T(\mathcal{C}) \end{array}$$

where v sends the pair (L_1, L_2) of excisive functors to the $(1, 1)$ -excisive functor $\mathcal{S}_{\text{fin},*}^2 \rightarrow \mathcal{C}$ given by

$$(X, Y) \mapsto L_1(X \wedge Y) \times_{L_1(*)=L_2(*)} L_2(Y).$$

PROOF. As in the proof of Lemma 8.34, our goal is to show that this diagram is a homotopy pullback in the Joyal model structure on simplicial sets, and the argument for that lemma also shows that $T(p)$ is a fibration, so it is sufficient to show that the induced map

$$f : \text{Exc}(\mathcal{S}_{\text{fin},*}, \mathcal{C}) \times_{\mathcal{C}} \text{Exc}(\mathcal{S}_{\text{fin},*}, \mathcal{C}) \rightarrow \mathcal{C} \times_{T\mathcal{C}} T^2(\mathcal{C})$$

is an equivalence of ∞ -categories.

Using Lemma 8.4 again, we can write f as a map

$$f : \text{Exc}(\mathcal{S}_{\text{fin},*}^2, \mathcal{C}) \rightarrow \text{Exc}^{1,1*}(\mathcal{S}_{\text{fin},*}^2, \mathcal{C}); \quad L \mapsto [(X, Y) \mapsto L(X \wedge Y, Y)]$$

from the ∞ -category of excisive functors $\mathcal{S}_{\text{fin},*}^2 \rightarrow \mathcal{C}$ to the ∞ -category of functors $M : \mathcal{S}_{\text{fin},*}^2 \rightarrow \mathcal{C}$ that are excisive in each variable individually and reduced in the

second variable (in the sense that the map $M(X, *) \xrightarrow{\sim} M(*, *)$ is an equivalence for each $X \in \mathcal{S}_{\text{fin},*}$).

To show that f is an equivalence of ∞ -categories, we describe an explicit homotopy inverse

$$g : \text{Exc}^{1,1*}(\mathcal{S}_{\text{fin},*}^2, \mathcal{C}) \rightarrow \text{Exc}(\mathcal{S}_{\text{fin},*}^2, \mathcal{C}).$$

For a functor $M : \mathcal{S}_{\text{fin},*}^2 \rightarrow \mathcal{C}$ we set

$$(8.37) \quad g(M)(X, Y) := M(X, S^0) \times_{M(*, S^0)} M(*, Y)$$

where the map $M(*, Y) \rightarrow M(*, S^0)$ used to construct this pullback is induced by the null map $Y \rightarrow S^0$ (that maps every point in Y to the basepoint in S^0).

We should show that if M is excisive in each variable and reduced in Y , then $g(M)$ is excisive as a functor $\mathcal{S}_{\text{fin},*}^2 \rightarrow \mathcal{C}$. Consider a pushout square in $\mathcal{S}_{\text{fin},*}^2$

$$\begin{array}{ccc} (X, Y) & \longrightarrow & (X_1, Y_1) \\ \downarrow & & \downarrow \\ (X_2, Y_2) & \longrightarrow & (X_0, Y_0) \end{array}$$

consisting of individual pushout squares in each variable. Applying $g(M)$ to this pushout square we get the square in \mathcal{C} given by

$$\begin{array}{ccc} M(X, S^0) \times_{M(*, S^0)} M(*, Y) & \longrightarrow & M(X_1, S^0) \times_{M(*, S^0)} M(*, Y_1) \\ \downarrow & & \downarrow \\ M(X_2, S^0) \times_{M(*, S^0)} M(*, Y_2) & \longrightarrow & M(X_0, S^0) \times_{M(*, S^0)} M(*, Y_0) \end{array}$$

Since M is excisive in each variable, this is a pullback of pullback squares, and hence is itself a pullback. Therefore $g(M)$ is excisive.

We then have, for $L \in \text{Exc}(\mathcal{S}_{\text{fin},*}^2, \mathcal{C})$:

$$\begin{aligned} g(f(L))(X, Y) &= f(L)(X, S^0) \times_{f(L)(* , S^0)} f(L)(* , Y) \\ &= L(X \wedge S^0, S^0) \times_{L(* \wedge S^0, S^0)} L(* \wedge Y, Y) \\ &\simeq L(X, S^0) \times_{L(*, S^0)} L(*, Y). \end{aligned}$$

Consider the diagram

$$\begin{array}{ccccccc} L(X, Y) & \longrightarrow & L(X, *) & \longrightarrow & L(X, S^0) & \longrightarrow & L(X, *) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ L(*, Y) & \longrightarrow & L(*, *) & \longrightarrow & L(*, S^0) & \longrightarrow & L(*, *) \end{array}$$

The first and third squares are pullbacks because L is excisive. The composite of the second and third squares is also a pullback. Hence the second square is a pullback by [Lur09a, 4.4.2.1], and so the composite of the first and second squares is also a pullback, again by [Lur09a, 4.4.2.1]. This calculation implies that the natural map

$$L \rightarrow g(f(L))$$

is an equivalence.

On the other hand we have, for $M \in \text{Exc}^{1,1*}(\mathcal{S}_{\text{fin},*}^2, \mathcal{C})$:

$$\begin{aligned} f(g(M))(X, Y) &= g(M)(X \wedge Y, Y) \\ &= M(X \wedge Y, S^0) \times_{M(*, S^0)} M(*, Y). \end{aligned}$$

We claim that $f(g(M))$ is equivalent to M . First note that M factors via the pointed ∞ -category $\mathcal{C}_{M(*,*)}$, of objects over/under $M(*, *)$. We can therefore assume without loss of generality that \mathcal{C} is pointed and that $M(*, *)$ is a null object. Since the map $M(*, Y) \rightarrow M(*, S^0)$ is then null, we can write $f(g(M))$ as the product

$$(8.38) \quad f(g(M))(X, Y) \simeq \text{hofib}[M(X \wedge Y, S^0) \rightarrow M(*, S^0)] \times M(*, Y).$$

Now observe that $M \in \text{Exc}^{1,1*}(\mathcal{S}_{\text{fin},*}^2, \mathcal{C})$ can be viewed as a functor

$$\mathcal{S}_{\text{fin},*} \rightarrow \text{Exc}_*(\mathcal{S}_{\text{fin},*}, \mathcal{C}); \quad X \mapsto M(X, -)$$

from the *pointed* ∞ -category $\mathcal{S}_{\text{fin},*}$ to the *stable* ∞ -category $\text{Exc}_*(\mathcal{S}_{\text{fin},*}, \mathcal{C}) \simeq \text{Sp}(\mathcal{C})$ of linear functors $\mathcal{S}_{\text{fin},*} \rightarrow \mathcal{C}$. Any such functor splits off the image of the null object, so we have an equivalence

$$(8.39) \quad M(X, Y) \simeq \text{hofib}[M(X, Y) \rightarrow M(*, Y)] \times M(*, Y).$$

Comparing (8.38) and (8.39) we see that to get an equivalence $f(g(M)) \simeq M$, it is enough to produce an equivalence

$$D(X, Y) \simeq D(X \wedge Y, S^0)$$

where $D : \mathcal{S}_{\text{fin},*}^2 \rightarrow \mathcal{C}$ is given by

$$D(X, Y) := \text{hofib}(M(X, Y) \rightarrow M(*, Y)).$$

This D is given by reducing M in its first variable, and it follows that D is reduced and excisive in both variables, i.e. is multilinear. The equivalence we need is a consequence of the classification of multilinear functors $\mathcal{S}_{\text{fin},*}^2 \rightarrow \mathcal{C}$ for any ∞ -category \mathcal{C} with finite limits.

To be explicit, it follows from [Lur17, 1.4.2.22] that the evaluation map

$$\text{Exc}_{*,*}^{1,1}(\mathcal{S}_{\text{fin},*}^2, \mathcal{C}) \rightarrow \text{Exc}_*(\mathcal{S}_{\text{fin},*}, \mathcal{C}); \quad D \mapsto D(-, S^0)$$

is an equivalence of ∞ -categories, and it is clear that a one-sided inverse is given by

$$\text{Exc}_*(\mathcal{S}_{\text{fin},*}, \mathcal{C}) \rightarrow \text{Exc}_{*,*}^{1,1}(\mathcal{S}_{\text{fin},*}^2, \mathcal{C}); \quad F \mapsto F(X \wedge Y).$$

Thus this functor is a two-sided inverse, and we therefore get the desired natural equivalence

$$D(X, Y) \simeq D(X \wedge Y, S^0).$$

This completes the proof that $M \simeq f(g(M))$, and hence the proof that f is an equivalence of ∞ -categories. \square

The Goodwillie Tangent Structure: Formal Construction

The constructions of Chapter 8 contain the basic data and lemmas on which our tangent structure rests, but to have a tangent ∞ -category we need to extend those definitions to an actual functor (i.e. map of simplicial sets)

$$T : \text{Weil} \times \text{Cat}_\infty^{\text{diff}} \rightarrow \text{Cat}_\infty^{\text{diff}}$$

which provides a strict action of the simplicial monoid Weil on the simplicial set $\text{Cat}_\infty^{\text{diff}}$. The goal of this chapter is to construct such a map. We start by giving an explicit description of the ∞ -category $\text{Cat}_\infty^{\text{diff}}$.

The ∞ -category of differentiable ∞ -categories. We recall Lurie's model for the ∞ -category of ∞ -categories from [Lur09a, 3.0.0.1].

DEFINITION 9.1. Let Cat_∞ be the simplicial nerve [Lur09a, 1.1.5.5] of the simplicial category whose objects are the ∞ -categories, and for which the simplicial mapping spaces are the maximal Kan complexes inside the usual functor ∞ -categories:

$$\text{Hom}_{\text{Cat}_\infty}(\mathcal{C}, \mathcal{D}) := \text{Fun}(\mathcal{C}, \mathcal{D})^\simeq,$$

i.e. the subcategory of $\text{Fun}(\mathcal{C}, \mathcal{D})$ whose morphisms are the natural equivalences. An n -simplex in Cat_∞ therefore consists of the following data:

- a sequence of ∞ -categories $\mathcal{C}_0, \dots, \mathcal{C}_n$;
- for each $0 \leq i < j \leq n$, a map of simplicial sets

$$\lambda_{i,j} : \mathcal{P}_{i,j} \rightarrow \text{Fun}(\mathcal{C}_i, \mathcal{C}_j)$$

where $\mathcal{P}_{i,j}$ denotes the poset of those subsets of

$$\{i, i+1, \dots, j-1, j\}$$

that include both i and j , ordered by inclusion;

such that

- (1) for each edge e in $\mathcal{P}_{i,j}$, the natural transformation $\lambda_{i,j}(e)$ is an equivalence;
- (2) for each $i < j < k$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{P}_{i,j} \times \mathcal{P}_{j,k} & \xrightarrow{\lambda_{i,j} \times \lambda_{j,k}} & \text{Fun}(\mathcal{C}_i, \mathcal{C}_j) \times \text{Fun}(\mathcal{C}_j, \mathcal{C}_k) \\ \cup \downarrow & & \downarrow \circ \\ \mathcal{P}_{i,k} & \xrightarrow{\lambda_{i,k}} & \text{Fun}(\mathcal{C}_i, \mathcal{C}_k). \end{array}$$

In particular, a 2-simplex in Cat_∞ comprises three functors

$$\begin{array}{ccc} \mathcal{C}_0 & \xrightarrow{H} & \mathcal{C}_2 \\ & \searrow F & \nearrow G \\ & & \mathcal{C}_1 \end{array}$$

together with a natural equivalence $H \xrightarrow{\sim} GF$.

DEFINITION 9.2. Let $\text{Cat}_\infty^{\text{diff}} \subseteq \text{Cat}_\infty$ be the maximal simplicial subset whose objects are the differentiable ∞ -categories and whose morphisms are the functors that preserve sequential colimits. We refer to $\text{Cat}_\infty^{\text{diff}}$ as the *∞ -category of differentiable ∞ -categories*.

Note that $\text{Cat}_\infty^{\text{diff}}$ is the simplicial nerve of a simplicial category whose objects are the differentiable ∞ -categories with simplicial mapping objects

$$\text{Hom}_{\text{Cat}_\infty^{\text{diff}}}(\mathcal{C}, \mathcal{D}) = \text{Fun}_{\mathbb{N}}(\mathcal{C}, \mathcal{D})^{\simeq},$$

the maximal Kan complex of the full subcategory $\text{Fun}_{\mathbb{N}}(\mathcal{C}, \mathcal{D}) \subseteq \text{Fun}(\mathcal{C}, \mathcal{D})$ of sequential-colimit-preserving functors. As is our convention, we do not distinguish notationally between the ∞ -category $\text{Cat}_\infty^{\text{diff}}$ and this underlying simplicial category.

Differentiable relative ∞ -categories. It turns out that $\text{Cat}_\infty^{\text{diff}}$ is not the most convenient ∞ -category on which to build the Goodwillie tangent structure. In this section, we describe another ∞ -category, denoted $\text{RelCat}_\infty^{\text{diff}}$, which is equivalent to $\text{Cat}_\infty^{\text{diff}}$ and which is more amenable.

The objects of $\text{RelCat}_\infty^{\text{diff}}$ are *relative ∞ -categories*. A *relative category* is simply a category \mathcal{C} together with a subcategory \mathcal{W} of ‘weak equivalences’ which contains all isomorphisms in \mathcal{C} . Associated to the pair $(\mathcal{C}, \mathcal{W})$ is an ∞ -category $\mathcal{C}[\mathcal{W}^{-1}]$ given by formally inverting the morphisms in \mathcal{W} . Barwick and Kan showed in [BK12] that any ∞ -category can be obtained this way, so that relative categories are yet another model for ∞ -categories.

Mazel-Gee [MG19] uses a nerve construction of Rezk to extend the localization construction to ‘relative ∞ -categories’ in which \mathcal{C} and \mathcal{W} are themselves allowed to be ∞ -categories. In other words, the Rezk nerve describes a very general ‘calculus of fractions’ for ∞ -categories.

The reason that relative ∞ -categories are convenient for us is that we can replace the ∞ -category $\text{Exc}^A(\mathcal{S}_{\text{fin},*}^n, \mathcal{C})$ of A -excisive functors with the *relative ∞ -category*

$$(\text{Fun}(\mathcal{S}_{\text{fin},*}^n, \mathcal{C}), P_A \mathcal{E})$$

consisting of the full ∞ -category of functors together with the subcategory consisting of those natural transformations that become equivalences on applying the A -excisive approximation P_A .

The benefit of this approach is that we can work directly with the functor ∞ -categories $\text{Fun}(\mathcal{S}_{\text{fin},*}^n, \mathcal{C})$ without involving the explicit P_A -approximation functor. In particular, it makes the monoidal nature of our construction almost immediate.

DEFINITION 9.3. A *relative ∞ -category* is a pair $(\mathcal{C}, \mathcal{W})$ consisting of an ∞ -category \mathcal{C} and a subcategory $\mathcal{W} \subseteq \mathcal{C}$ that includes all equivalences in \mathcal{C} . (In particular \mathcal{W} contains all the objects of \mathcal{C} .)

A *relative functor* $G : (\mathcal{C}_0, \mathcal{W}_0) \rightarrow (\mathcal{C}_1, \mathcal{W}_1)$ between relative ∞ -categories is a functor $G : \mathcal{C}_0 \rightarrow \mathcal{C}_1$ such that $G(\mathcal{W}_0) \subseteq \mathcal{W}_1$.

A *natural transformation* between relative functors $G, G' : (\mathcal{C}_0, \mathcal{W}_0) \rightarrow (\mathcal{C}_1, \mathcal{W}_1)$ is a natural transformation between the functors $G, G' : \mathcal{C}_0 \rightarrow \mathcal{C}_1$, that is, a functor $\alpha : \Delta^1 \times \mathcal{C}_0 \rightarrow \mathcal{C}_1$ that restricts to G on $\{0\} \times \mathcal{C}_0$ and to G' on $\{1\} \times \mathcal{C}_0$.

We say that a natural transformation α is a *relative equivalence* if for each $X \in \mathcal{C}_0$, the morphism $\alpha_X : G(X) \rightarrow G'(X)$ is in the subcategory $\mathcal{W}_1 \subseteq \mathcal{C}_1$. In this case, we also say that the natural transformation α *takes values in* \mathcal{W}_1 .

EXAMPLE 9.4. Associated to any ∞ -category is a *minimal* relative ∞ -category $(\mathcal{C}, \mathcal{E}_e)$ where \mathcal{E}_e is the subcategory of equivalences in \mathcal{C} . A relative functor between minimal relative ∞ -categories is just a functor between the underlying ∞ -categories, and a relative equivalence between such functors is just a natural equivalence in the usual sense.

We now wish to restrict to those relative ∞ -categories whose localization is a *differentiable* ∞ -category. In fact, it will be convenient to consider not all such relative ∞ -categories, but only those for which the localization is an exact reflective subcategory.

DEFINITION 9.5. We say that a relative ∞ -category $(\mathcal{C}, \mathcal{W})$ is *differentiable* if \mathcal{C} is a differentiable ∞ -category, and \mathcal{W} is the subcategory of local equivalences for an exact localization functor $\mathcal{C} \rightarrow \mathcal{C}$ in the sense of [Lur09a, 5.2.7]. In other words, there exists an adjunction of differentiable ∞ -categories

$$f : \mathcal{C} \rightleftarrows \mathcal{D} : g$$

such that

- f preserves finite limits;
- g is fully faithful and preserves sequential colimits;
- \mathcal{W} is the subcategory of f -equivalences in \mathcal{C} , i.e. those morphisms that are mapped by f to an equivalence in \mathcal{D} .

We will say that a relative functor $G : (\mathcal{C}_0, \mathcal{W}_0) \rightarrow (\mathcal{C}_1, \mathcal{W}_1)$ between differentiable relative ∞ -categories is *differentiable* if its underlying functor $G : \mathcal{C}_0 \rightarrow \mathcal{C}_1$ preserves sequential colimits.

Our task is now to build an ∞ -category whose objects are the differentiable relative ∞ -categories, and which is equivalent to $\mathbb{C}\text{at}_\infty^{\text{diff}}$. We start by giving a simplicial enrichment to the category of differentiable relative ∞ -categories and differentiable relative functors.

DEFINITION 9.6. Let $\mathbb{R}\text{el}_0\mathbb{C}\text{at}_\infty^{\text{diff}}$ be the simplicial category in which:

- objects are the differentiable relative ∞ -categories $(\mathcal{C}, \mathcal{W})$;
- the simplicial mapping object $\text{Hom}_{\mathbb{R}\text{el}_0\mathbb{C}\text{at}_\infty^{\text{diff}}}((\mathcal{C}_0, \mathcal{W}_0), (\mathcal{C}_1, \mathcal{W}_0))$ is given by the subcategory

$$\text{Fun}_{\mathbb{N}}^{\sim}((\mathcal{C}_0, \mathcal{W}_0), (\mathcal{C}_1, \mathcal{W}_0)) \subseteq \text{Fun}(\mathcal{C}_0, \mathcal{C}_1)$$

whose objects are the differentiable relative functors, and whose morphisms are the relative equivalences.

PROPOSITION 9.7. *There is a Dwyer-Kan equivalence of simplicial categories (i.e. an equivalence in Bergner's model structure [Ber07])*

$$M_0 : \mathbb{C}\text{at}_\infty^{\text{diff}} \rightarrow \mathbb{R}\text{el}_0\mathbb{C}\text{at}_\infty^{\text{diff}}$$

that sends each differentiable ∞ -category \mathcal{C} to the differentiable relative ∞ -category $(\mathcal{C}, \mathcal{E}_{\mathcal{C}})$ where $\mathcal{E}_{\mathcal{C}}$ is the subcategory of equivalences in \mathcal{C} , and on simplicial mapping objects is given by the equivalences (in fact, equalities):

$$\mathrm{Fun}_{\mathbb{N}}^{\sim}(\mathcal{C}_0, \mathcal{C}_1) \xrightarrow{\sim} \mathrm{Fun}_{\mathbb{N}}^{\sim}((\mathcal{C}_0, \mathcal{E}_{\mathcal{C}_0}), (\mathcal{C}_1, \mathcal{E}_{\mathcal{C}_1})).$$

PROOF. First note that when \mathcal{C} is a differentiable ∞ -category, $(\mathcal{C}, \mathcal{E}_{\mathcal{C}})$ is a differentiable relative ∞ -category; the identity adjunction on \mathcal{C} satisfies the conditions of Definition 9.5.

Since M_0 is clearly fully faithful, it remains to show that it is essentially surjective on objects. So let $(\mathcal{C}, \mathcal{W})$ be an arbitrary differentiable relative ∞ -category. Then we know that \mathcal{W} is the subcategory of f -equivalences for some adjunction of differentiable ∞ -categories

$$f : \mathcal{C} \rightleftarrows \mathcal{D} : g$$

that satisfies the conditions of Definition 9.5. The functors f and g both preserve sequential colimits, so determine differentiable relative functors

$$f : (\mathcal{C}, \mathcal{W}) \rightleftarrows (\mathcal{D}, \mathcal{E}_{\mathcal{D}}) : g$$

which we claim are isomorphisms in the homotopy category of the simplicial category $\mathbb{R}\mathrm{el}_0\mathrm{Cat}_{\infty}^{\mathrm{diff}}$.

The counit of the localizing adjunction (f, g) is an equivalence $\epsilon : fg \xrightarrow{\sim} 1_{\mathcal{D}}$ and therefore also a relative equivalence $fg \xrightarrow{\sim} 1_{(\mathcal{D}, \mathcal{E}_{\mathcal{D}})}$. Hence $fg = 1_{(\mathcal{D}, \mathcal{E}_{\mathcal{D}})}$ in the homotopy category of $\mathbb{R}\mathrm{el}_0\mathrm{Cat}_{\infty}^{\mathrm{diff}}$.

It remains to produce a relative equivalence between gf and $1_{(\mathcal{C}, \mathcal{W})}$. The unit of the localizing adjunction (f, g) is a natural transformation $1_{\mathcal{C}} \rightarrow gf$ between relative functors $(\mathcal{C}, \mathcal{W}) \rightarrow (\mathcal{C}, \mathcal{W})$. To test if this natural transformation is a relative equivalence, we must show that it becomes an equivalence in \mathcal{D} after applying f . But $f\eta : f \rightarrow fgf$ is inverse to the equivalence ϵf , so this is indeed the case. Hence $gf = 1_{(\mathcal{C}, \mathcal{W})}$ in the homotopy category too, and so f and g are isomorphisms as claimed. Thus M_0 is essentially surjective. \square

Note that $\mathbb{R}\mathrm{el}_0\mathrm{Cat}_{\infty}^{\mathrm{diff}}$ is enriched in ∞ -categories but not in Kan complexes because a relative equivalence is not necessarily invertible. The simplicial nerve of $\mathbb{R}\mathrm{el}_0\mathrm{Cat}_{\infty}^{\mathrm{diff}}$ is therefore not an ∞ -category. In order to rectify this problem, we add inverses for the relative equivalences by taking a fibrant replacement for the simplicial mapping objects in $\mathbb{R}\mathrm{el}_0\mathrm{Cat}_{\infty}^{\mathrm{diff}}$.

For this purpose we use an explicit fibrant replacement for the Quillen model structure given by Kan's Ex^{∞} functor [Kan57]. We do not give a complete definition of the functor $\mathrm{Ex}^{\infty} : \mathrm{Set}_{\Delta} \rightarrow \mathrm{Set}_{\Delta}$, but here are the key properties from our point of view. For any simplicial set Y , the simplicial set $\mathrm{Ex}^{\infty}(Y)$ is a fibrant replacement for Y in the Quillen model structure. So $\mathrm{Ex}^{\infty}(Y)$ is a Kan complex, and there is a natural map $r_Y : Y \xrightarrow{\sim} \mathrm{Ex}^{\infty}(Y)$ which is a weak equivalence and cofibration in that model structure.

The vertices of $\mathrm{Ex}^{\infty}(Y)$ can be identified with the vertices of Y , and the edges of $\mathrm{Ex}^{\infty}(Y)$ can be identified with zigzags

$$y_0 \rightarrow y_1 \leftarrow y_2 \rightarrow \cdots \leftarrow y_{2k}$$

of edges in Y , where we identify zigzags of different lengths by including additional identity morphisms on the right. The map r sends an edge $y_0 \rightarrow y_1$ of Y to the

zigzag

$$y_0 \rightarrow y_1 = y_1.$$

The functor Ex^∞ preserves finite products and has simplicial enrichment coming from the composite

$$X \times \text{Ex}^\infty(Y) \xrightarrow{r_X} \text{Ex}^\infty(X) \times \text{Ex}^\infty(Y) \cong \text{Ex}^\infty(X \times Y).$$

When Y is an ∞ -category, we can think of the Kan complex $\text{Ex}^\infty(Y)$ as a model for the ‘ ∞ -groupoidification’ of Y given by freely inverting the 1-simplexes.

DEFINITION 9.8. Let $\mathbb{R}\text{elCat}_\infty^{\text{diff}}$ denote the simplicial category whose objects are the differentiable relative ∞ -categories, and whose simplicial mapping spaces are the Kan complexes

$$\text{Hom}_{\mathbb{R}\text{elCat}_\infty^{\text{diff}}}((\mathcal{C}_0, \mathcal{W}_0), (\mathcal{C}_1, \mathcal{W}_1)) := \text{Ex}^\infty \text{Fun}_{\mathbb{N}}^{\sim}((\mathcal{C}_0, \mathcal{W}_0), (\mathcal{C}_1, \mathcal{W}_1))$$

given by applying Ex^∞ to the simplicial mapping objects in $\mathbb{R}\text{el}_0\text{Cat}_\infty^{\text{diff}}$. Composition in $\mathbb{R}\text{elCat}_\infty^{\text{diff}}$ is induced by that in $\mathbb{R}\text{el}_0\text{Cat}_\infty^{\text{diff}}$ using the fact that Ex^∞ preserves finite products.

There is a canonical functor

$$r : \mathbb{R}\text{el}_0\text{Cat}_\infty^{\text{diff}} \rightarrow \mathbb{R}\text{elCat}_\infty^{\text{diff}}$$

given by the identity on objects and by inclusions of the form $r_Y : Y \rightarrow \text{Ex}^\infty(Y)$ on mapping spaces. Since r_Y is a weak equivalence in the Quillen model structure, r is a Dwyer-Kan equivalence.

By construction $\mathbb{R}\text{elCat}_\infty^{\text{diff}}$ is enriched in Kan complexes, and hence its simplicial nerve is an ∞ -category which, following our usual convention, we also denote $\mathbb{R}\text{elCat}_\infty^{\text{diff}}$.

COROLLARY 9.9. *There is an equivalence of ∞ -categories*

$$M : \text{Cat}_\infty^{\text{diff}} \xrightarrow{\sim} \mathbb{R}\text{elCat}_\infty^{\text{diff}}$$

given by composing the map M_0 from Proposition 9.7 with the functor

$$r : \mathbb{R}\text{el}_0\text{Cat}_\infty^{\text{diff}} \xrightarrow{\sim} \mathbb{R}\text{elCat}_\infty^{\text{diff}}$$

of Definition 9.8.

PROOF. Each of M_0 and r is a Dwyer-Kan equivalence, so their composite is a Dwyer-Kan equivalence between fibrant objects in the Bergner model structure. Taking simplicial nerves we get an equivalence of ∞ -categories. \square

The equivalence M tells us that we can use $\mathbb{R}\text{elCat}_\infty^{\text{diff}}$ as a model for the ∞ -category of differentiable ∞ -categories and sequential-colimit-preserving functors. We show in the next section that $\mathbb{R}\text{elCat}_\infty^{\text{diff}}$ admits the required tangent structure, which can then be transferred along M to $\text{Cat}_\infty^{\text{diff}}$ using Lemma 3.12.

We conclude this section by giving an explicit description of the simplexes in the ∞ -category $\mathbb{R}\text{elCat}_\infty^{\text{diff}}$ which will be useful in constructing our tangent structure.

REMARK 9.10. An n -simplex λ in the ∞ -category $\mathbb{R}\text{elCat}_\infty^{\text{diff}}$ consists of the following data:

- a sequence of differentiable relative ∞ -categories

$$(\mathcal{C}_0, \mathcal{W}_0), \dots, (\mathcal{C}_n, \mathcal{W}_n);$$

- for each $0 \leq i < j \leq n$, a functor

$$\lambda_{i,j} : \mathcal{P}_{i,j} \rightarrow \mathrm{Ex}^\infty \mathrm{Fun}(\mathcal{C}_i, \mathcal{C}_j)$$

subject to the following conditions:

- (1) for each object $I \in \mathcal{P}_{i,j}$, $\lambda_{i,j}(I) : \mathcal{C}_i \rightarrow \mathcal{C}_j$ is a differentiable relative functor;
- (2) for each morphism $\iota : I \subseteq I'$ in $\mathcal{P}_{i,j}$, the edge

$$\lambda_{i,j}(\iota) \in \mathrm{Ex}^\infty \mathrm{Fun}(\mathcal{C}_i, \mathcal{C}_j)_1$$

is a zigzag of relative equivalences in $\mathrm{Fun}(\mathcal{C}_i, \mathcal{C}_j)$;

- (3) for each $i < j < k$, the following diagram commutes

$$\begin{array}{ccc} \mathcal{P}_{i,j} \times \mathcal{P}_{j,k} & \xrightarrow{\lambda_{i,j} \times \lambda_{j,k}} & \mathrm{Ex}^\infty \mathrm{Fun}(\mathcal{C}_i, \mathcal{C}_j) \times \mathrm{Ex}^\infty \mathrm{Fun}(\mathcal{C}_j, \mathcal{C}_k) \\ \downarrow \cup & & \downarrow \mathrm{Ex}^\infty(\circ) \\ \mathcal{P}_{i,k} & \xrightarrow{\lambda_{i,k}} & \mathrm{Ex}^\infty \mathrm{Fun}(\mathcal{C}_i, \mathcal{C}_k). \end{array}$$

Tangent structure on differentiable relative ∞ -categories. We can now build a tangent structure on the ∞ -category $\mathrm{RelCat}_\infty^{\mathrm{diff}}$ by describing explicitly the corresponding action map

$$T : \mathrm{Weil} \times \mathrm{RelCat}_\infty^{\mathrm{diff}} \rightarrow \mathrm{RelCat}_\infty^{\mathrm{diff}}.$$

In order to make this a strict action of the monoidal ∞ -category Weil , we need to be careful about one point. When we write a functor ∞ -category of the form

$$\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C})$$

we will actually mean the isomorphic simplicial set

$$\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}, \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}, \dots, \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}) \dots))$$

with n iterations. It follows that the simplicial sets $\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^m, \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C}))$ and $\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{m+n}, \mathcal{C})$ are actually *equal* not merely isomorphic.

With that warning in mind, we start by defining our desired functor T on objects.

DEFINITION 9.11. Let A be a Weil-algebra with n generators, and $(\mathcal{C}, \mathcal{W})$ a differentiable relative ∞ -category. We define a relative ∞ -category

$$T^A(\mathcal{C}, \mathcal{W}) := (\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C}), P_A \mathcal{W})$$

where $P_A \mathcal{W}$ is the subcategory of the functor ∞ -category $\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C})$ consisting of those morphisms, i.e. natural transformations $\beta : \Delta^1 \times \mathcal{S}_{\mathrm{fin},*}^n \rightarrow \mathcal{C}$, that, after applying the A -excisive approximation functor P_A from Definition 8.5, take values in \mathcal{W} . That is, for each $X \in \mathcal{S}_{\mathrm{fin},*}^n$, the component $(P_A \beta)_X$ is in \mathcal{W} .

LEMMA 9.12. *The relative ∞ -category $T^A(\mathcal{C}, \mathcal{W})$ is differentiable.*

PROOF. Since \mathcal{C} is differentiable, and limits and colimits in a functor ∞ -category are calculated objectwise, it follows that $\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C})$ is also differentiable. It therefore remains to show that $P_A \mathcal{W}$ is determined by a suitable localizing adjunction.

Since $(\mathcal{C}, \mathcal{W})$ is a differentiable relative ∞ -category, there is an adjunction of differentiable ∞ -categories

$$f : \mathcal{C} \rightleftarrows \mathcal{D} : g$$

satisfying the conditions of Definition 9.5. Now consider the pair of adjunctions

$$\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C}) \begin{array}{c} \xrightarrow{P_A} \\ \iota \end{array} \mathrm{Exc}^A(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C}) \begin{array}{c} \xrightarrow{f_*} \\ g_* \end{array} \mathrm{Exc}^A(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{D})$$

where the maps f_* and g_* are given by composing with f and g respectively, noting that since these functors preserve finite limits, f_* and g_* preserve A -excisive functors.

We verify that the composed adjunction $(f_*P_A, \iota g_*)$ satisfies the conditions of Definition 9.5:

- P_A preserves finite limits by Proposition 8.5, and f_* preserves finite limits because f does and those limits are calculated objectwise in $\mathrm{Exc}^A(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C})$ and $\mathrm{Exc}^A(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{D})$ (also by 8.5);
- ι is fully faithful and preserves sequential colimits by 8.5; g_* is fully faithful because g is, and because $\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^n, -)$ preserves fully faithful inclusion; g_* preserves sequential colimits again because of 8.5;
- a morphism β in $\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C})$ is an f_*P_A -equivalence if and only if $P_A(\beta)$ is an f_* -equivalence if and only if (since equivalences in the ∞ -category $\mathrm{Exc}^A(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{D})$ are detected objectwise) $P_A(\beta)_X$ is an f -equivalence, i.e. in \mathcal{W} , for all $X \in \mathcal{S}_{\mathrm{fin},*}^n$; thus the subcategory $P_A\mathcal{W}$ consists precisely of the f_*P_A -equivalences.

Thus $(\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C}), P_A\mathcal{W})$ is a differentiable relative ∞ -category as claimed. \square

Next, we define T on morphisms.

DEFINITION 9.13. Let $\phi : A_0 \rightarrow A_1$ be a labelled morphism of Weil-algebras, and let $G : (\mathcal{C}_0, \mathcal{W}_0) \rightarrow (\mathcal{C}_1, \mathcal{W}_1)$ be a differentiable relative functor between differentiable relative ∞ -categories.

We let $T^\phi(G) : (\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n_0}, \mathcal{C}_0), P_{A_0}\mathcal{W}_0) \rightarrow (\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n_1}, \mathcal{C}_1), P_{A_1}\mathcal{W}_1)$ be the relative functor

$$T^\phi(G) : \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n_0}, \mathcal{C}_0) \rightarrow \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n_1}, \mathcal{C}_1); \quad L \mapsto GL\tilde{\phi}$$

given by composition with the maps of simplicial sets $\tilde{\phi} : \mathcal{S}_{\mathrm{fin},*}^{n_1} \rightarrow \mathcal{S}_{\mathrm{fin},*}^{n_0}$ (of Definition 8.12) and $G : \mathcal{C}_0 \rightarrow \mathcal{C}_1$.

LEMMA 9.14. *The functor $T^\phi(G)$ is a differentiable relative functor*

$$T^{A_0}(\mathcal{C}_0, \mathcal{W}_0) \rightarrow T^{A_1}(\mathcal{C}_1, \mathcal{W}_1).$$

PROOF. We must show that $T^\phi(G)(P_{A_0}\mathcal{W}_0) \subseteq P_{A_1}\mathcal{W}_1$, so consider a morphism of $\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n_0}, \mathcal{C}_0)$, i.e. a natural transformation $L \rightarrow L'$, such that the induced map $(P_{A_0}L)(X) \rightarrow (P_{A_0}L')(X)$ is in \mathcal{W}_0 for all $X \in \mathcal{S}_{\mathrm{fin},*}^{n_0}$.

First, since G is a relative functor, it follows that

$$G(P_{A_0}L)\tilde{\phi}(Y) \rightarrow G(P_{A_0}L)\tilde{\phi}(Y)$$

is in \mathcal{W}_1 for any $Y \in \mathcal{S}_{\mathrm{fin},*}^{n_1}$.

Now recall that since $(\mathcal{C}_1, \mathcal{W}_1)$ is differentiable, there is an adjunction

$$f_1 : \mathcal{C}_1 \rightleftarrows \mathcal{D}_1 : g_1$$

satisfying the conditions of Definition 9.5. Thus \mathcal{W}_1 is the subcategory of f_1 -equivalences, and so the map

$$f_1G(P_{A_0}L)\tilde{\phi} \rightarrow f_1G(P_{A_0}L')\tilde{\phi}$$

is a natural equivalence between functors $\mathcal{S}_{\text{fin},*}^{n_1} \rightarrow \mathcal{D}_1$. Since \mathcal{D}_1 is differentiable, we can apply P_{A_1} to this map to obtain a natural equivalence

$$P_{A_1}(f_1G(P_{A_0}L)\tilde{\phi}) \xrightarrow{\sim} P_{A_1}(f_1G(P_{A_0}L')\tilde{\phi}).$$

Since f_1 preserves both sequential colimits and finite limits, it commutes with the construction P_{A_1} by Lemma 8.6. Thus we also have a natural equivalence

$$f_1P_{A_1}(G(P_{A_0}L)\tilde{\phi}) \xrightarrow{\sim} f_1P_{A_1}(G(P_{A_0}L')\tilde{\phi}).$$

In other words, the natural map

$$P_{A_1}(G(P_{A_0}L)\tilde{\phi}) \rightarrow P_{A_1}(G(P_{A_0}L')\tilde{\phi})$$

takes values in \mathcal{W}_1 . But, applying the equivalences of (8.11) and (8.23), it follows that

$$P_{A_1}(GL\tilde{\phi}) \rightarrow P_{A_1}(GL'\tilde{\phi})$$

takes values in \mathcal{W}_1 , so that

$$GL\tilde{\phi} \rightarrow GL'\tilde{\phi}$$

is in $P_{A_1}\mathcal{W}_1$. So $T^\phi(G)$ is a relative functor as required.

Finally, $T^\phi(G)$ preserves sequential colimits because G does and these colimits are calculated objectwise in the functor ∞ -categories. So $T^\phi(G)$ is differentiable. \square

Before moving on to the general case, it is worthwhile also to define T explicitly on 2-simplexes.

DEFINITION 9.15. Let ϕ be a 2-simplex in Weil consisting of, in part, a pair of labelled Weil-algebra morphisms

$$A_0 \xrightarrow{\phi_1} A_1 \xrightarrow{\phi_2} A_2$$

and a labelling on the composite $\phi_2\phi_1$.

Let λ be a 2-simplex in $\mathbb{R}\text{elCat}_\infty^{\text{diff}}$. According to Remark 9.10, λ consists of a diagram of relative functors

$$\begin{array}{ccc} (\mathcal{C}_0, \mathcal{W}_0) & \xrightarrow{H} & (\mathcal{C}_2, \mathcal{W}_2) \\ & \searrow F & \nearrow G \\ & & (\mathcal{C}_1, \mathcal{W}_1) \end{array}$$

together with an edge in $\text{Ex}^\infty \text{Fun}(\mathcal{C}_0, \mathcal{C}_2)$, that is, a zigzag

$$\lambda_{0,1,2} : H \rightarrow E_1 \leftarrow \cdots \rightarrow E_{2k-1} \leftarrow GF,$$

in which each map is a relative equivalence.

We define $T^\phi(\lambda)$ to be the 2-simplex in $\mathbb{R}\text{elCat}_\infty^{\text{diff}}$ consisting of the corresponding diagram of relative functors

$$\begin{array}{ccc} T^{A_0}(\mathcal{C}_0, \mathcal{W}_0) & \xrightarrow{T^{\phi_2\phi_1}(H)} & T^{A_2}(\mathcal{C}_2, \mathcal{W}_2) \\ & \searrow T^{\phi_1}(F) & \nearrow T^{\phi_2}(G) \\ & & T^{A_1}(\mathcal{C}_1, \mathcal{W}_1) \end{array}$$

together with the following zigzag of relative equivalences

$$(9.16) \quad (H(-)\widetilde{\phi_2\phi_1}) \rightarrow (E_1(-)\widetilde{\phi_1\phi_2}) \leftarrow \cdots \rightarrow (E_{2k-1}(-)\widetilde{\phi_1\phi_2}) \leftarrow (GF(-)\widetilde{\phi_1\phi_2})$$

between $T^{\phi_2\phi_1}(H)$ and $T^{\phi_2}(G)T^{\phi_1}(F)$, in which each map is induced by the corresponding map in $\lambda_{0,1,2}$, and the first map, in addition, involves the natural transformation

$$\alpha : \widetilde{\phi_2\phi_1} \rightarrow \widetilde{\phi_1\phi_2}$$

associated to the 2-simplex ϕ by Definition 8.18.

LEMMA 9.17. *The construction of $T^\phi(\lambda)$ in Definition 9.15 produces a 2-simplex in $\mathbb{R}\text{elCat}_\infty^{\text{diff}}$.*

PROOF. The only thing left to check is that each of the natural transformations in the zigzag (9.16) is a relative equivalence, i.e. that each of these natural transformations takes values in $P_{A_2}\mathcal{W}_2$, that is, takes values in \mathcal{W}_2 after applying P_{A_2} .

Let $\gamma : E \rightarrow E'$ be a relative equivalence between relative functors

$$E, E' : (\mathcal{C}_0, \mathcal{W}_0) \rightarrow (\mathcal{C}_2, \mathcal{W}_2),$$

i.e. for each $X \in \mathcal{C}_0$, the map $\gamma_X : E(X) \rightarrow E'(X)$ is in \mathcal{W}_2 .

It follows that for every functor $L : \mathcal{S}_{\text{fin},*}^{n_0} \rightarrow \mathcal{C}_0$, the map induced by γ

$$EL\widetilde{\phi_1\phi_2} \rightarrow E'L\widetilde{\phi_1\phi_2}$$

takes values in \mathcal{W}_2 . A similar argument to that in the proof of Lemma 9.14 implies that

$$P_{A_2}(EL\widetilde{\phi_1\phi_2}) \rightarrow P_{A_2}(E'L\widetilde{\phi_1\phi_2})$$

takes values in \mathcal{W}_2 . Therefore the map induced by γ ,

$$E(-)\widetilde{\phi_1\phi_2} \rightarrow E'(-)\widetilde{\phi_1\phi_2}$$

takes values in $P_{A_2}\mathcal{W}_2$ as required.

This argument shows that each map in the zigzag (9.16) after the first is a relative equivalence. To show that the first map is also a relative equivalence, we note that

$$P_{A_2}(E_1(-)\widetilde{\phi_2\phi_1}) \rightarrow P_{A_2}(E_1(-)\widetilde{\phi_1\phi_2})$$

is an equivalence of the type described in (8.21), hence is in \mathcal{W}_2 . Combined with the previous argument, we deduce that

$$H(-)\widetilde{\phi_2\phi_1} \rightarrow E_1(-)\widetilde{\phi_1\phi_2}$$

is also a relative equivalence. □

We now extend our constructions above to simplexes of arbitrary dimension.

DEFINITION 9.18. Let ϕ be an n -simplex in $\mathbb{W}\text{eil}$ consisting of, in part, a sequence of labelled Weil-algebra morphisms

$$A_0 \xrightarrow{\phi_1} A_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_n} A_n$$

together with compatible labellings on all composites.

Also let λ be an n -simplex in $\mathbb{R}\text{elCat}_\infty^{\text{diff}}$ as described in Remark 9.10.

We define $T^\phi(\lambda)$ to be the n -simplex in $\mathbb{R}\text{elCat}_\infty^{\text{diff}}$ consisting of the differentiable relative ∞ -categories

$$T^{A_0}(\mathcal{C}_0, \mathcal{W}_0), \dots, T^{A_n}(\mathcal{C}_n, \mathcal{W}_n)$$

and the functors

$$T^\phi(\lambda)_{i,j} : \mathcal{P}_{i,j} \rightarrow \text{Ex}^\infty \text{Fun}(\text{Fun}(\mathcal{S}_{\text{fin},*}^{n_i}, \mathcal{C}_i), \text{Fun}(\mathcal{S}_{\text{fin},*}^{n_j}, \mathcal{C}_j))$$

constructed as follows.

Firstly, from the n -simplex ϕ , we construct for $0 \leq i < j \leq n$ a functor

$$\tilde{\phi}_{i,j} : \mathcal{P}_{i,j} \rightarrow \text{Fun}(\mathcal{S}_{\text{fin},*}^{n_j}, \mathcal{S}_{\text{fin},*}^{n_i}).$$

An object of $\mathcal{P}_{i,j}$ is a subset

$$I = \{i = i_0, i_1, \dots, i_k = j\} \subseteq \{i, i+1, \dots, j-1, j\}$$

and we let $\tilde{\phi}_{i,j}(I)$ be the composite functor

$$(\phi_{i_k} \cdots \phi_{i_{k-1}+1}) \cdots (\phi_{i_1} \cdots \phi_{i_0+1}) : \mathcal{S}_{\text{fin},*}^{n_j} \rightarrow \mathcal{S}_{\text{fin},*}^{n_i}.$$

For a morphism e in $\mathcal{P}_{i,j}$, that is an inclusion $I \subseteq I'$ of subsets of $\{i, i+1, \dots, j-1, j\}$ that include i and j , we have to produce a natural transformation

$$\tilde{\phi}_{i,j}(e) : \tilde{\phi}_{i,j}(I) \rightarrow \tilde{\phi}_{i,j}(I') : \mathcal{S}_{\text{fin},*}^{n_j} \rightarrow \mathcal{S}_{\text{fin},*}^{n_i}.$$

For example, when e is the inclusion $\{0, 2\} \subseteq \{0, 1, 2\}$, then $\tilde{\phi}_{0,2}(e)$ is required to be a natural transformation

$$\widetilde{\phi_2 \phi_1} \rightarrow \tilde{\phi}_1 \tilde{\phi}_2$$

which we choose to be the map α of Definition 8.18. For a general morphism e , the desired map $\tilde{\phi}_{i,j}(e)$ is obtained by (a composite of) generalizations of α to more than two factors. It follows from Lemma 8.32 that, for inclusions $I \xrightarrow{e} I' \xrightarrow{e'} I''$, we have

$$\tilde{\phi}_{i,j}(e'e) = \tilde{\phi}_{i,j}(e')\tilde{\phi}_{i,j}(e)$$

and so we obtain a functor $\tilde{\phi}_{i,j} : \mathcal{P}_{i,j} \rightarrow \text{Fun}(\mathcal{S}_{\text{fin},*}^{n_j}, \mathcal{S}_{\text{fin},*}^{n_i})$ as desired.

From the n -simplex λ , we also have a functor

$$\lambda_{i,j} : \mathcal{P}_{i,j} \rightarrow \text{Ex}^\infty \text{Fun}(\mathcal{C}_i, \mathcal{C}_j)$$

as described in Remark 9.10.

We complete the definition of the n -simplex $T^\phi(\lambda)$ by defining the functor $T^\phi(\lambda)_{i,j}$ to be the composite

$$\begin{aligned} \mathcal{P}_{i,j} &\xrightarrow{\langle \tilde{\phi}_{i,j}, \lambda_{i,j} \rangle} \text{Fun}(\mathcal{S}_{\text{fin},*}^{n_j}, \mathcal{S}_{\text{fin},*}^{n_i}) \times \text{Ex}^\infty \text{Fun}(\mathcal{C}_i, \mathcal{C}_j) \\ &\xrightarrow{\langle r, \text{Id} \rangle} \text{Ex}^\infty \text{Fun}(\mathcal{S}_{\text{fin},*}^{n_j}, \mathcal{S}_{\text{fin},*}^{n_i}) \times \text{Ex}^\infty \text{Fun}(\mathcal{C}_i, \mathcal{C}_j) \\ &\xrightarrow{\cong} \text{Ex}^\infty (\text{Fun}(\mathcal{S}_{\text{fin},*}^{n_j}, \mathcal{S}_{\text{fin},*}^{n_i}) \times \text{Fun}(\mathcal{C}_i, \mathcal{C}_j)) \\ &\xrightarrow{\text{Ex}^\infty(c)} \text{Ex}^\infty \text{Fun}(\text{Fun}(\mathcal{S}_{\text{fin},*}^{n_i}, \mathcal{C}_i), \text{Fun}(\mathcal{S}_{\text{fin},*}^{n_j}, \mathcal{C}_j)) \end{aligned}$$

where

$$c : \text{Fun}(\mathcal{S}_{\text{fin},*}^{n_j}, \mathcal{S}_{\text{fin},*}^{n_i}) \times \text{Fun}(\mathcal{C}_i, \mathcal{C}_j) \rightarrow \text{Fun}(\text{Fun}(\mathcal{S}_{\text{fin},*}^{n_i}, \mathcal{C}_i), \text{Fun}(\mathcal{S}_{\text{fin},*}^{n_j}, \mathcal{C}_j))$$

is adjoint to the composition map.

LEMMA 9.19. *The construction of Definition 9.18 defines an n -simplex $T^\phi(\lambda)$ in $\mathbb{R}elCat_\infty^{\text{diff}}$.*

PROOF. We will verify each of the conditions in Remark 9.10. First note that each $T^{A_i}(\mathcal{C}_i, \mathcal{W}_i)$ is a differentiable relative ∞ -category by Lemma 9.12.

Now consider an object $I = \{i = i_0, i_1, \dots, i_k = j\} \in \mathcal{P}_{i,j}$. Then $T^\phi(\lambda)_{i,j}(I)$ is the functor

$$\text{Fun}(\mathcal{S}_{\text{fin},*}^{n_i}, \mathcal{C}_i) \rightarrow \text{Fun}(\mathcal{S}_{\text{fin},*}^{n_j}, \mathcal{C}_j)$$

given by pre-composition with the functor

$$(\phi_{i_k} \cdots \widetilde{\phi_{i_{k-1}+1}}) \cdots (\phi_{i_1} \cdots \widetilde{\phi_{i_0+1}}) : \mathcal{S}_{\text{fin},*}^{n_j} \rightarrow \mathcal{S}_{\text{fin},*}^{n_i}$$

and post-composition with the differentiable relative functor $\lambda_{i,j}(I) : \mathcal{C}_i \rightarrow \mathcal{C}_j$. The argument of Lemma 9.14, with $\tilde{\phi}$ replaced by the composite functor

$$(\phi_{i_k} \cdots \widetilde{\phi_{i_{k-1}+1}}) \cdots (\phi_{i_1} \cdots \widetilde{\phi_{i_0+1}}),$$

and G replaced by $\lambda_{i,j}(I)$, implies that $T^\phi(\lambda)_{i,j}(I)$ is a differentiable relative functor. This verifies condition (1) of Remark 9.10 for our proposed n -simplex $T^\phi(\lambda)$.

Next consider an edge $I \subseteq I'$ in $\mathcal{P}_{i,j}$. Then $T^\phi(\lambda)_{i,j}$ applied to that edge is a zigzag of natural transformations

$$T^\phi(\lambda)_{i,j}(I) \rightarrow \cdots \leftarrow T^\phi(\lambda)_{i,j}(I') : \text{Fun}(\mathcal{S}_{\text{fin},*}^{n_i}, \mathcal{C}_i) \rightarrow \text{Fun}(\mathcal{S}_{\text{fin},*}^{n_j}, \mathcal{C}_j)$$

induced by natural transformations of the type $\alpha : \widetilde{\phi_2 \phi_1} \rightarrow \tilde{\phi}_1 \tilde{\phi}_2$ and a zigzag $\lambda_{i,j}(I) \rightarrow \cdots \leftarrow \lambda_{i,j}(I')$. The argument of Lemma 9.17, slightly generalized, implies that each entry in this zigzag is a relative equivalence. This verifies condition (2) of Remark 9.10.

It remains to check condition (3), which is a large but easy diagram-chase in the category of simplicial sets, and which follows from the corresponding condition for the n -simplex λ , the naturality of Ex^∞ , and the fact that the following diagrams involving the functors $\tilde{\phi}_{i,j}$ commute:

$$\begin{array}{ccc} \mathcal{P}_{i,j} \times \mathcal{P}_{j,k} & \xrightarrow{\tilde{\phi}_{i,j} \times \tilde{\phi}_{j,k}} & \text{Fun}(\mathcal{S}_{\text{fin},*}^{n_j}, \mathcal{S}_{\text{fin},*}^{n_i}) \times \text{Fun}(\mathcal{S}_{\text{fin},*}^{n_k}, \mathcal{S}_{\text{fin},*}^{n_j}) \\ \downarrow \cup & & \downarrow \circ \\ \mathcal{P}_{i,k} & \xrightarrow{\tilde{\phi}_{i,k}} & \text{Fun}(\mathcal{S}_{\text{fin},*}^{n_k}, \mathcal{S}_{\text{fin},*}^{n_i}). \end{array}$$

□

PROPOSITION 9.20. *The construction of T on simplexes in Definition 9.18 gives a well-defined action of the simplicial monoid Weil on the simplicial set $\mathbb{R}elCat_\infty^{\text{diff}}$.*

PROOF. The definition commutes with the simplicial structure so defines a map of simplicial sets

$$T : \text{Weil} \times \mathbb{R}elCat_\infty^{\text{diff}} \rightarrow \mathbb{R}elCat_\infty^{\text{diff}}.$$

To show that the map T is a strict action map, consider first the case of 0-simplexes. We have to check that

$$T^{A'} T^A(\mathcal{C}, \mathcal{W}) = T^{A' \otimes A}(\mathcal{C}, \mathcal{W}).$$

Recall that we have chosen our functor ∞ -categories so that there is an *equality* (not just an isomorphism)

$$\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n'}, \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C})) = \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n+n'}, \mathcal{C})$$

so it is sufficient to show that the subcategories $P_{A'}(P_A\mathcal{W})$ and $P_{A' \otimes A}\mathcal{W}$ are equal.

In other words, let $f : L \rightarrow L'$ be a natural transformation between functors $\mathcal{S}_{\mathrm{fin},*}^{n+n'} \rightarrow \mathcal{C}$. Then f is in $P_{A'}(P_A\mathcal{W})$ if $P_{A'}L \rightarrow P_{A'}L'$ takes values in $P_A\mathcal{W}$, i.e. if

$$P_AP_{A'}L \rightarrow P_AP_{A'}L'$$

takes values in \mathcal{W} . Since $P_AP_{A'} \simeq P_{A' \otimes A}$, this is the case if and only if f is in $P_{A' \otimes A}\mathcal{W}$.

Now let us turn to higher degree simplexes. We have to show that, for n -simplexes ϕ, ϕ' in Weil , and an n -simplex λ in $\mathbb{R}\mathrm{elCat}_{\infty}^{\mathrm{diff}}$, we have

$$T^{\phi' \otimes \phi}(\lambda) = T^{\phi'}T^{\phi}(\lambda).$$

That is, we have to show that two functors

$$\mathcal{P}_{i,j} \rightarrow \mathrm{Ex}^{\infty} \mathrm{Fun}(\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n'_i+n_i}, \mathcal{C}_i), \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n'_j+n_j}, \mathcal{C}_j))$$

are equal. This is another large diagram-chase; the key fact is that the following diagram commutes:

$$\begin{array}{ccc} & \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n'_j}, \mathcal{S}_{\mathrm{fin},*}^{n'_i}) \times \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n_j}, \mathcal{S}_{\mathrm{fin},*}^{n_i}) & \\ \langle \tilde{\phi}'_{i,j}, \tilde{\phi}_{i,j} \rangle \nearrow & \downarrow \cong \times & \\ \mathcal{P}_{i,j} & & \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n'_j+n_j}, \mathcal{S}_{\mathrm{fin},*}^{n'_i+n_i}). \\ \langle \widetilde{\phi' \otimes \phi} \rangle_{i,j} \searrow & & \end{array}$$

□

We now transfer the Weil-action on $\mathbb{R}\mathrm{elCat}_{\infty}^{\mathrm{diff}}$ along the equivalence of ∞ -categories $M : \mathrm{Cat}_{\infty}^{\mathrm{diff}} \xrightarrow{\sim} \mathbb{R}\mathrm{elCat}_{\infty}^{\mathrm{diff}}$ of Corollary 9.9, using Lemma 3.12.

DEFINITION 9.21. Let $T : \mathrm{Weil}^{\otimes} \rightarrow \mathrm{End}(\mathrm{Cat}_{\infty}^{\mathrm{diff}})^{\circ}$ be the monoidal functor given by composing the adjoint of the action map of Proposition 9.20 with the monoidal equivalence

$$\mathrm{End}(\mathbb{R}\mathrm{elCat}_{\infty}^{\mathrm{diff}})^{\circ} \simeq \mathrm{End}(\mathrm{Cat}_{\infty}^{\mathrm{diff}})^{\circ}$$

associated to the equivalence M of Corollary 9.9, as given by Lemma 3.12.

We finally have the following result (completing the definition of the Goodwillie tangent structure and the proof of Theorem 7.15).

THEOREM 9.22. *The monoidal functor*

$$T : \mathrm{Weil}^{\otimes} \rightarrow \mathrm{End}(\mathrm{Cat}_{\infty}^{\mathrm{diff}})^{\circ}$$

of Definition 9.21 determines a cartesian tangent structure on the ∞ -category $\mathrm{Cat}_{\infty}^{\mathrm{diff}}$, in the sense of Definition 3.2, whose underlying endofunctor and projection are, up to equivalence, as described in 7.12 and following.

PROOF. To see that T is a tangent structure on $\mathbb{C}at_{\infty}^{\text{diff}}$, we have to show that it preserves the foundational and vertical lift pullbacks. This claim follows from Lemma 8.34 and Proposition 8.36 with one proviso; we must show that the homotopy pullbacks in the Joyal model structure appearing in those results are pullbacks in the ∞ -category $\mathbb{C}at_{\infty}^{\text{diff}}$.

We prove that claim by applying a result of Riehl and Verity [RV22, 6.3.12] with $\mathcal{K} = \mathbb{C}AT_{\infty}$ the ∞ -cosmos of ∞ -categories. (See [RV22, Ch. 1] for an introduction to the theory of ∞ -cosmoses.) We deduce from that result that there is a ‘cosmologically-embedded’ sub- ∞ -cosmos $\mathbb{C}AT_{\infty}^{\mathbb{N}} \subseteq \mathbb{C}AT_{\infty}$ whose objects are the ∞ -categories that admit sequential colimits, and whose 1-morphisms are the functors that preserve sequential colimits.

The claim that $\mathbb{C}AT_{\infty}^{\mathbb{N}}$ is cosmologically-embedded [RV22, 6.3.3] implies that any square diagram in $\mathbb{C}AT_{\infty}^{\mathbb{N}}$ that is a pullback along a fibration in $\mathbb{C}AT_{\infty}$ is also a pullback along a fibration in $\mathbb{C}AT_{\infty}^{\mathbb{N}}$. The pullbacks of 8.34 and 8.36 fit that bill, and so they determine pullbacks in the corresponding ∞ -category $\mathbb{C}at_{\infty}^{\mathbb{N}}$ (of ∞ -categories that admit sequential colimits and functors that preserve them), and hence also in the full subcategory $\mathbb{C}at_{\infty}^{\text{diff}} \subseteq \mathbb{C}at_{\infty}^{\mathbb{N}}$.

Finally, it follows from Lemma 8.35 that $\mathbb{C}at_{\infty}^{\text{diff}}$ has finite products which are preserved by the tangent bundle functor, so the tangent structure T is cartesian. \square

Differential Objects are Stable ∞ -Categories

Having constructed the Goodwillie tangent structure, we now turn to its initial study, and in this chapter we look at its differential objects. Since the objects of $\text{Cat}_\infty^{\text{diff}}$ are *differentiable* ∞ -categories, there is a serious danger of confusing the words ‘differential’ and ‘differentiable’ in this chapter.

Recall from [Lur17, 1.1.3.4] that an ∞ -category \mathcal{C} is *stable* if it is pointed, admits finite limits and colimits, and a square in \mathcal{C} is a pushout if and only if it is a pullback. (In particular, a stable ∞ -category admits biproducts which we denote by \oplus .) Also recall from [Lur17, 6.1.1.7] that a stable ∞ -category is differentiable if and only if it admits countable coproducts. We then have the following simple characterization.

THEOREM 10.1. *A differentiable ∞ -category \mathcal{C} admits a differential structure within the Goodwillie tangent structure if and only if \mathcal{C} is a stable ∞ -category.*

PROOF. We apply Corollary 5.26, so it is sufficient to show that the tangent spaces in $\text{Cat}_\infty^{\text{diff}}$ are the stable ∞ -categories. For any differentiable ∞ -category \mathcal{C} and object $X \in \mathcal{C}$, we can identify $T_X \mathcal{C}$ with the ∞ -category of excisive functors $\mathcal{S}_{\text{fin},*} \rightarrow \mathcal{C}$ that map $*$ to X . We therefore have

$$T_X \mathcal{C} \simeq \text{Exc}_*(\mathcal{S}_{\text{fin},*}, \mathcal{C}_{/X})$$

where the right-hand side is the ∞ -category of *reduced* excisive functors from $\mathcal{S}_{\text{fin},*}$ to the slice ∞ -category $\mathcal{C}_{/X}$. Thus $T_X \mathcal{C}$ is stable by [Lur17, 1.4.2.16].

Conversely, if \mathcal{C} is stable, then

$$T_* \mathcal{C} \simeq \text{Exc}_*(\mathcal{S}_{\text{fin},*}, \mathcal{C})$$

which is equivalent to \mathcal{C} by [Lur17, 1.4.2.21]. Therefore \mathcal{C} is equivalent to a tangent space and so admits a differential structure by 5.26. \square

The last part of this proof provides a canonical identification of any stable (differentiable) ∞ -category \mathcal{C} with a tangent space, and hence, by Proposition 5.24, a canonical differential structure on each such \mathcal{C} . This observation allows us to define a cartesian differential structure (see [BCS09]) on the homotopy category of stable (differentiable) ∞ -categories.

THEOREM 10.2. *Let $\text{Cat}_\infty^{\text{diff},\text{st}}$ be the full subcategory of $\text{Cat}_\infty^{\text{diff}}$ whose objects are the stable differentiable ∞ -categories. Then the homotopy category $h\text{Cat}_\infty^{\text{diff},\text{st}}$ has a cartesian differential structure in which the monoid structure on an object \mathcal{C} is given by the biproduct functor $\oplus : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, and the derivative of a morphism $F : \mathcal{C} \rightarrow \mathcal{D}$ is the ‘directional derivative’*

$$\nabla(F) : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{D}$$

given by

$$\widehat{\nabla}(F)(V, X) := D_1(F(X \oplus -))(V).$$

This formula denotes Goodwillie's linear approximation D_1 applied to the functor $F(X \oplus -) : \mathcal{C} \rightarrow \mathcal{D}$ and evaluated at object $V \in \mathcal{C}$.

PROOF. It is not hard to verify directly that the axioms in [BCS09, 2.1.1] hold for $h\text{Cat}_\infty^{\text{diff, st}}$, but we deduce this theorem from the results of Chapter 5 in order to illustrate how the cartesian differential structure on $h\text{Cat}_\infty^{\text{diff, st}}$ is related to the Goodwillie tangent structure on $\text{Cat}_\infty^{\text{diff}}$.

Recall that Theorem 5.29 determines a cartesian differential structure on the category $\widehat{h\text{Diff}}(\text{Cat}_\infty^{\text{diff}})$ whose objects are the differential objects in $\text{Cat}_\infty^{\text{diff}}$, and whose morphisms are maps in $h\text{Cat}_\infty^{\text{diff}}$ between underlying objects.

We define a functor

$$T_* : h\text{Cat}_\infty^{\text{diff, st}} \rightarrow \widehat{h\text{Diff}}(\text{Cat}_\infty^{\text{diff}})$$

by sending the stable differentiable ∞ -category \mathcal{C} to the differential object in $\text{Cat}_\infty^{\text{diff}}$ with underlying object the tangent space

$$T_*\mathcal{C} = \text{Exc}_*(\mathcal{S}_{\text{fin}, *}, \mathcal{C}) = \mathcal{S}p(\mathcal{C})$$

and differential structure determined by Proposition 5.24. For a functor $F : \mathcal{C} \rightarrow \mathcal{D}$, we define $T_*(F)$ to be the morphism in $h\text{Cat}_\infty^{\text{diff}}$ given by the composite

$$T_*\mathcal{C} \xrightarrow[\sim]{\Omega^\infty} \mathcal{C} \xrightarrow{F} \mathcal{D} \xleftarrow[\sim]{\Omega^\infty} T_*\mathcal{D}$$

where $\Omega^\infty : T_*\mathcal{C} = \mathcal{S}p(\mathcal{C}) \xrightarrow{\sim} \mathcal{C}$ denotes evaluation of a reduced excisive functor at S^0 ; see [Lur17, 1.4.2.21]. This definition makes T_* into a fully faithful embedding since morphisms in both categories are taken from $h\text{Cat}_\infty^{\text{diff}}$.

Given an object of $\widehat{h\text{Diff}}(\text{Cat}_\infty^{\text{diff}})$, i.e. a differential object \mathcal{D} in the Goodwillie tangent structure, we know from Theorem 10.1 that the underlying ∞ -category of \mathcal{D} is stable. The equivalence

$$\Omega^\infty : T_*\mathcal{D} \xrightarrow{\sim} \mathcal{D}$$

is an isomorphism in $\widehat{h\text{Diff}}(\text{Cat}_\infty^{\text{diff}})$, so T_* is essentially surjective on objects. Hence T_* is an equivalence of categories, and we can transfer the cartesian differential structure from Theorem 5.29 along T_* to $h\text{Cat}_\infty^{\text{diff, st}}$.

It remains to show that this inherited cartesian differential structure is as claimed in the statement of the theorem. To calculate that structure, we first examine the differential structure on the tangent space $T_*\mathcal{C}$.

Working through the construction of the functor \mathcal{J}_\bullet in Proposition 5.24, we see that the E_∞ -monoid structure on $T_*\mathcal{C}$ is given by restricting the 'addition' in the tangent bundle $T\mathcal{C}$, i.e. is the objectwise product

$$+ : T_*\mathcal{C} \times T_*\mathcal{C} \rightarrow T_*\mathcal{C}; \quad (L_1, L_2) \mapsto L_1(-) \times L_2(-).$$

Since this map commutes with evaluation at S^0 , we deduce that the corresponding monoid structure on \mathcal{C} is also the product (and hence biproduct)

$$\oplus : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}.$$

The map $\hat{p} : T(T_*\mathcal{C}) \rightarrow T_*\mathcal{C}$ associated to the differential structure on $T_*\mathcal{C}$ is precisely the map g appearing in the proof of Proposition 8.36, that is,

$$\hat{p}(M)(X) = \text{hofib}[M(X, S^0) \rightarrow M(*, S^0)]$$

where $M : \mathcal{S}_{\text{fin},*}^2 \rightarrow \mathcal{C}$ is excisive in both variables and reduced in its second variable. Transferring back to \mathcal{C} along Ω^∞ , we deduce that $\hat{p} : T(\mathcal{C}) \rightarrow \mathcal{C}$ is given by

$$\hat{p}(L) := \text{hofib}[L(S^0) \rightarrow L(*)].$$

The derivative $\nabla(F)$ of a morphism $F : \mathcal{C} \rightarrow \mathcal{D}$ in $h\text{Cat}_\infty^{\text{diff, st}}$ is the composite

$$\mathcal{C} \times \mathcal{C} \xleftarrow[\sim]{\langle p, \hat{p} \rangle} T(\mathcal{C}) \xrightarrow{T(F)} T(\mathcal{D}) \xrightarrow{\hat{p}} \mathcal{D}.$$

To evaluate this composite at $(X, V) \in \mathcal{C} \times \mathcal{C}$ we have to identify an excisive functor $L : \mathcal{S}_{\text{fin},*} \rightarrow \mathcal{C}$ such that $L(*) \simeq X$ and $\text{hofib}[L(S^0) \rightarrow L(*)] \simeq V$. We can express this functor as

$$L(-) = X \oplus (- \otimes V)$$

where \otimes denotes the canonical tensoring of a pointed ∞ -category \mathcal{C} with finite colimits over $\mathcal{S}_{\text{fin},*}$. (The functor $\otimes : \mathcal{S}_{\text{fin},*} \times \mathcal{C} \rightarrow \mathcal{C}$ can be constructed using the characterization in [Lur17, 1.4.2.6] of the ∞ -category \mathcal{S}_{fin} of finite spaces.) It follows that

$$\nabla(F)(X, V) = \text{hofib}[P_1(F(X \oplus (- \otimes V)))(S^0) \rightarrow F(X)].$$

Since $- \otimes V$ commutes with colimits, it also commutes with the construction of P_1 , so that we have

$$\nabla(F)(X, V) = \text{hofib}[P_1(F(X \oplus -))(S^0 \otimes V) \rightarrow F(X)]$$

which is precisely $D_1(F(X \oplus -))(V)$ as claimed. \square

Theorem 10.2 is closely related to [BJO⁺18, Cor. 6.6], which is the result that first inspired this paper. Let us briefly discuss the connection.

DEFINITION 10.3. Let $h\text{Cat}^{\text{ab}}$ be the category in which:

- an object is an abelian category;
- a morphism from \mathcal{A} to \mathcal{B} is a pointwise-chain-homotopy class of functors

$$F : \mathcal{A} \rightarrow \text{Ch}_+(\mathcal{B})$$

where the target is the category of non-negatively-graded chain complexes of objects in \mathcal{B} , and two such functors F, G are *pointwise-chain-homotopy equivalent* if for each object $A \in \mathcal{A}$ there is a chain-homotopy equivalence $F(A) \simeq G(A)$.

Composition of morphisms is achieved via ‘Dold-Kan prolongation’ of such an F to a functor $\text{Ch}_+(\mathcal{A}) \rightarrow \text{Ch}_+(\mathcal{B})$, see [BJO⁺18, 3.2].

THEOREM 10.4 (Bauer-Johnson-Osborne-Riehl-Tebbe [BJO⁺18, 6.6]). *The category $h\text{Cat}^{\text{ab}}$ has a cartesian differential structure in which the monoid structure on an object \mathcal{A} is given by the biproduct $\oplus : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ and the derivative of a morphism $F : \mathcal{A} \rightarrow \text{Ch}_+(\mathcal{B})$ is the ‘directional derivative’ $\nabla(F) : \mathcal{A} \times \mathcal{A} \rightarrow \text{Ch}_+(\mathcal{B})$ given by*

$$\nabla(F)(X, V) := D_1(F(X \oplus -))(V)$$

where D_1 denotes the linearization of a chain-complex-valued functor in the sense of Johnson and McCarthy [JM04].

Despite the close similarity, there are slight differences between the context of Theorems 10.4 and 10.2 that prohibit a direct comparison. In particular, note that $h\text{Cat}^{\text{ab}}$ is defined using *pointwise* chain-homotopy equivalence, rather than natural equivalence (though we suspect that [BJO⁺18] could have been written entirely in terms of natural chain-homotopy equivalence instead).

Modulo that distinction, we speculate that there is an equivalence of cartesian differential categories between a subcategory of $h\text{Cat}^{\text{ab}}$ (say, on those abelian categories \mathcal{A} that admit countable coproducts and suitably continuous functors) with a subcategory of $h\text{Cat}_{\infty}^{\text{diff, st}}$ (say, on the corresponding stable ∞ -categories $N_{dg}\text{Ch}(\mathcal{A})$ given by the differential graded nerves [Lur17, 1.3.2.10] of the categories of chain complexes on such \mathcal{A}). We do not pursue a precise equivalence of this form here.

Jets and n -Excisive Functors

Goodwillie’s notion of excisive functor played a central role in the construction of what we have called the Goodwillie tangent structure on the ∞ -category $\text{Cat}_\infty^{\text{diff}}$ of differentiable ∞ -categories. Our goal in this chapter is to show that the notions of n -excisive functor, for $n > 1$, are implicit in that tangent structure, so that the entirety of Goodwillie’s theory can be recovered from it.

We actually describe how the notion of n -excisive *equivalence*, i.e. the condition that a natural transformation determines an equivalence between n -excisive approximations, relates to the Goodwillie tangent structure. The notion from ordinary differential geometry that corresponds to n -excisive equivalence is that of ‘ n -jet’. Recall that we say two smooth maps $f, g : M \rightarrow N$ between smooth manifolds *agree to order n at $x \in M$* if $f(x) = g(x)$, and the (multivariable) Taylor series of f and g in local coordinates at x agree up to degree n . The *n -jet at x* of the map f is its equivalence class under the relation of agreeing to order n at x .

We can interpret the Taylor series condition in terms of the standard tangent structure on the category Mfld . Let $T_x^n(M)$ denote the *n -fold tangent space to M at x* , that is, the fibre of the projection map $T^n(M) \rightarrow M$ over the point x , where $T^n(M)$ is the n -fold iterate of the tangent bundle functor T . A smooth map $f : M \rightarrow N$ then induces a smooth map

$$T_x^n(f) : T_x^n(M) \rightarrow T_{f(x)}^n(N) \subseteq T^n(N)$$

i.e. the restriction of $T^n(f)$ to $T_x^n(M)$.

LEMMA 11.1. *Let $f, g : M \rightarrow N$ be smooth maps. Then f and g have Taylor series at x that agree to degree n if and only if $T_x^n(f) = T_x^n(g)$.*

The main result of this chapter is an analogue of Lemma 11.1 that connects the higher excisive approximations in Goodwillie calculus to the Goodwillie tangent structure on $\text{Cat}_\infty^{\text{diff}}$ constructed in Chapter 9. We refer the reader to [Goo03] for the original theory of n -excisive approximation, and to [Lur17, 6.1.1] for the generalization of that theory to (differentiable) ∞ -categories.

DEFINITION 11.2. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a sequential-colimit-preserving functor between differentiable ∞ -categories, and suppose that \mathcal{C} admits finite colimits. For an object $X \in \mathcal{C}$, the *n -excisive approximation to F at X* is

$$P_n^X F := P_n(F_{/X}) : \mathcal{C}_{/X} \rightarrow \mathcal{D}$$

that is, the n -excisive approximation of the restriction of F to the slice ∞ -category $\mathcal{C}_{/X}$ of objects over X .

THEOREM 11.3. *Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be morphisms in $\text{Cat}_\infty^{\text{diff}}$, where \mathcal{C} admits finite colimits. Let $\alpha : F \rightarrow G$ be a natural transformation, and let X be an object*

of \mathcal{C} . Then α induces an equivalence

$$P_n^X \alpha : P_n^X F \xrightarrow{\sim} P_n^X G$$

of n -excisive approximations at X if and only if α induces an equivalence

$$T^n(\alpha)\iota_X : T^n(F)\iota_X \xrightarrow{\sim} T^n(G)\iota_X$$

in the functor ∞ -category

$$\mathrm{Fun}(T_X^n(\mathcal{C}), T^n(\mathcal{D})),$$

where $T_X^n(\mathcal{C})$ is the fibre over X of the projection $T^n(\mathcal{C}) \rightarrow \mathcal{C}$, and

$$\iota_X : T_X^n(\mathcal{C}) \rightarrow T^n(\mathcal{C})$$

is the inclusion of that fibre.

PROOF. We start by noting that each of the conditions in question implies that $\alpha_X : F(X) \xrightarrow{\sim} G(X)$ is an equivalence: if $P_n^X \alpha$ is an equivalence, then so is

$$P_0^X \alpha \simeq \alpha_X,$$

and if $T^n(\alpha)\iota_X$ is an equivalence, then so is

$$p_{\mathcal{D}}^n T^n(\alpha)\iota_X \simeq \alpha_X.$$

Replacing \mathcal{C} with $\mathcal{C}/_X$ and \mathcal{D} with $\mathcal{D}/_{G(X)}$, we can now reduce to the case that X is a terminal object in \mathcal{C} , and F, G are reduced.

In this case, we can identify the objects of $T_X^n(\mathcal{C})$ with the functors $L : S_{\mathrm{fin},*}^n \rightarrow \mathcal{C}$ that are excisive in each variable separately and satisfy $L(*, \dots, *) \simeq *$. Then

$$T_X^n(F) : T_X^n(\mathcal{C}) \rightarrow T^n(\mathcal{D})$$

corresponds to the map

$$L \mapsto P_{1,\dots,1}(FL).$$

Our goal is therefore to show that $\alpha : F \rightarrow G$ induces an equivalence

$$P_n \alpha : P_n F \rightarrow P_n G$$

if and only if it induces an equivalence

$$P_{1,\dots,1}(\alpha L) : P_{1,\dots,1}(FL) \rightarrow P_{1,\dots,1}(GL)$$

for every reduced and $(1, \dots, 1)$ -excisive functor $L : S_{\mathrm{fin},*}^n \rightarrow \mathcal{C}$.

Suppose first that $P_n \alpha$ is an equivalence, and consider the commutative diagram

$$(11.4) \quad \begin{array}{ccc} P_{1,\dots,1}(FL) & \longrightarrow & P_{1,\dots,1}((P_n F)L) \\ \downarrow & & \downarrow \\ P_{1,\dots,1}(P_n(FL)) & \longrightarrow & P_{1,\dots,1}(P_n((P_n F)L)) \end{array}$$

The vertical maps are given by n -excisive approximation and are equivalences since being $(1, \dots, 1)$ -excisive is a stronger condition than being n -excisive, by [Lur17, 6.1.3.4]. The bottom horizontal map is an equivalence by Lemma 7.8, so the top map is too. From the assumption that $P_n \alpha : P_n F \rightarrow P_n G$ is an equivalence, it therefore follows that $P_{1,\dots,1}(\alpha L)$ is an equivalence too.

Conversely suppose that $P_{1,\dots,1}(\alpha L)$ is an equivalence for any reduced functor $L : S_{\mathrm{fin},*}^n \rightarrow \mathcal{C}$ that is excisive in each variable. Note that this same condition then

holds for any reduced L regardless of it being excisive; that claim follows from the equivalences

$$P_{1,\dots,1}(FL) \xrightarrow{\sim} P_{1,\dots,1}(F(P_{1,\dots,1}L))$$

given by (8.11).

Our strategy for showing that $P_n\alpha$ is an equivalence is to use induction on the Taylor tower. Consider the pullback diagram

$$(11.5) \quad \begin{array}{ccc} P_k F & \longrightarrow & P_{k-1} F \\ \downarrow & & \downarrow \\ * & \longrightarrow & R_k F \end{array}$$

of [Lur17, 6.1.2.4] (Goodwillie's delooping theorem for homogeneous functors), in which $R_k F : \mathcal{C} \rightarrow \mathcal{D}$ is k -homogeneous. By induction on k , it suffices to show that α induces an equivalence $R_k\alpha : R_k F \rightarrow R_k G$ for each $1 \leq k \leq n$.

It is clear from (11.5) that the construction R_k naturally takes values in the ∞ -category \mathcal{D}_* of pointed objects in \mathcal{D} . By [Lur17, 6.1.2.11], it is sufficient to show that $R_k\alpha$ is an equivalence of k -homogeneous functors $\mathcal{C}_* \rightarrow \mathcal{D}_*$ between pointed ∞ -categories. Such functors naturally factor via $\mathcal{S}p(\mathcal{D}_*)$ by [Lur17, 6.1.2.9], so it is sufficient to show that $\Omega R_k\alpha$ is an equivalence, i.e. that α induces an equivalence

$$D_k\alpha : D_k F \rightarrow D_k G$$

between the k -th layers of the Taylor towers of functors $\mathcal{C}_* \rightarrow \mathcal{D}_*$.

To show that $D_k\alpha$ is an equivalence we show that α induces an equivalence on multilinearized cross-effects. Since \mathcal{C}_* is pointed and admits finite colimits, it has a canonical tensoring over $\mathcal{S}_{\text{fin},*}$, i.e. a functor

$$\otimes : \mathcal{S}_{\text{fin},*} \times \mathcal{C}_* \rightarrow \mathcal{C}_*$$

that preserves finite colimits in each variable, and such that $S^0 \otimes Y \simeq Y$ for each $Y \in \mathcal{C}_*$. The functor \otimes can be constructed from the characterization in [Lur17, 1.4.2.6] of the ∞ -category \mathcal{S}_{fin} of finite spaces.

Take objects $A_1, \dots, A_n \in \mathcal{C}_*$ and consider the functor

$$L : \mathcal{S}_{\text{fin},*}^n \rightarrow \mathcal{C}_*; \quad (X_1, \dots, X_n) \mapsto (X_1 \otimes A_1) \vee \dots \vee (X_n \otimes A_n)$$

where \vee is the coproduct in \mathcal{C}_* . Our hypothesis on α implies that $P_{1,\dots,1}(\alpha L)$ is an equivalence.

Since the functor

$$\mathcal{S}_{\text{fin},*}^n \rightarrow \mathcal{C}_*^n; \quad (X_1, \dots, X_n) \mapsto (X_1 \otimes A_1, \dots, X_n \otimes A_n)$$

preserves colimits (including the null object) in each variable, it commutes with $P_{1,\dots,1}$, by [Lur17, 6.1.1.30]. It follows that α induces an equivalence

$$P_{1,\dots,1}(\alpha(- \vee \dots \vee -))(- \otimes A_1, \dots, - \otimes A_n).$$

Evaluating at (S^0, \dots, S^0) , we deduce that α induces an equivalence (of functors $\mathcal{C}_*^n \rightarrow \mathcal{D}_*$):

$$(11.6) \quad P_{1,\dots,1}(F(- \vee \dots \vee -)) \xrightarrow{\sim} P_{1,\dots,1}(G(- \vee \dots \vee -)).$$

For any $1 \leq k \leq n$, the k -th cross-effect of F , see [Lur17, 6.1.3.20], is the total homotopy fibre of a k -cube whose entries are functors of the form

$$F(- \vee \dots \vee -)$$

with some subset of its arguments replaced by the null object $*$ in \mathcal{C}_* . Since $P_{1,\dots,1}$ commutes with the construction of that total homotopy fibre, it follows from (11.6) that α induces an equivalence

$$P_{1,\dots,1}(\mathrm{cr}_k F) \xrightarrow{\sim} P_{1,\dots,1}(\mathrm{cr}_k G)$$

for all $1 \leq k \leq n$. It follows by [Lur17, 6.1.3.23] that α then induces equivalences

$$\mathrm{cr}_k(D_k F) \rightarrow \mathrm{cr}_k(D_k G)$$

and hence also, by [Lur17, 6.1.4.7], equivalences

$$D_k F \rightarrow D_k G$$

for $1 \leq k \leq n$, as required. This completes the proof that $P_n \alpha : P_n F \xrightarrow{\sim} P_n G$ is an equivalence. \square

REMARK 11.7. Theorem 11.3 explains how the notion of n -excisive approximation is related to the Goodwillie tangent structure on $\mathcal{C}\mathrm{at}_\infty^{\mathrm{diff}}$. However, it is not quite true to say that this notion is fully encoded in that tangent structure, since the statement of Theorem 11.3 relies on natural transformations that are not equivalences, and hence are not part of the ∞ -category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{diff}}$. We can include those natural transformations by replacing $\mathcal{C}\mathrm{at}_\infty^{\mathrm{diff}}$ with a corresponding ∞ -bicategory $\mathcal{C}\mathrm{AT}_\infty^{\mathrm{diff}}$, which is the subject of the next chapter.

The $(\infty, 2)$ -Category of Differentiable ∞ -Categories

The goal of this chapter is to show that the Goodwillie tangent structure on $\mathcal{C}at_{\infty}^{\text{diff}}$ extends to a tangent structure, in the sense of Definition 6.16, on an ∞ -bicategory $\mathcal{C}AT_{\infty}^{\text{diff}}$ of differentiable ∞ -categories. We start by defining that object.

DEFINITION 12.1. Let $\mathcal{C}AT_{\infty}^{\text{diff}}$ be the scaled nerve (see Example 6.5) of the simplicial category whose objects are the differentiable ∞ -categories, with simplicial mapping objects given by the ∞ -categories

$$\text{Hom}_{\mathcal{C}AT_{\infty}^{\text{diff}}}(\mathcal{C}, \mathcal{D}) := \text{Fun}_{\mathbb{N}}(\mathcal{C}, \mathcal{D})$$

of sequential-colimit-preserving functors. Since each mapping object is an ∞ -category, $\mathcal{C}AT_{\infty}^{\text{diff}}$ is an ∞ -bicategory.

Our construction of a tangent structure on the ∞ -bicategory $\mathcal{C}AT_{\infty}^{\text{diff}}$ follows a similar path to that on the ∞ -category $\mathcal{C}at_{\infty}^{\text{diff}}$. We again start with relative differentiable ∞ -categories (Definition 7.4).

DEFINITION 12.2. Let $\mathbb{R}el_0\mathcal{C}AT_{\infty}^{\text{diff}}$ be the simplicial category in which:

- objects are the differentiable relative ∞ -categories $(\mathcal{C}, \mathcal{W})$;
- mapping simplicial sets are the full subcategories

$$\text{Fun}_{\mathbb{N}}((\mathcal{C}_0, \mathcal{W}_0), (\mathcal{C}_1, \mathcal{W}_1)) \subseteq \text{Fun}(\mathcal{C}_0, \mathcal{C}_1)$$

consisting of the differentiable relative functors, i.e. functors $G : \mathcal{C}_0 \rightarrow \mathcal{C}_1$ which preserve sequential colimits and for which $G(\mathcal{W}_0) \subseteq \mathcal{W}_1$.

The scaled nerve of $\mathbb{R}el_0\mathcal{C}AT_{\infty}^{\text{diff}}$ is an ∞ -bicategory by Example 6.5, but to get an accurate model for $\mathcal{C}AT_{\infty}^{\text{diff}}$ we still need to invert the relative equivalences in the simplicial mapping objects, just as we did in constructing the ∞ -category $\mathbb{R}el\mathcal{C}at_{\infty}^{\text{diff}}$ in Definition 9.6. We accomplish this inversion by forming the following homotopy pushout of ∞ -bicategories.

DEFINITION 12.3. Let $\mathbb{R}el_1\mathcal{C}AT_{\infty}^{\text{diff}}$ be the scaled simplicial set given by the pushout (in the category of scaled simplicial sets):

$$(12.4) \quad \begin{array}{ccc} \mathbb{R}el_0\mathcal{C}at_{\infty}^{\text{diff}} & \xrightarrow{r} & \mathbb{R}el\mathcal{C}at_{\infty}^{\text{diff}} \\ \downarrow & & \downarrow \\ \mathbb{R}el_0\mathcal{C}AT_{\infty}^{\text{diff}} & \longrightarrow & \mathbb{R}el_1\mathcal{C}AT_{\infty}^{\text{diff}} \end{array}$$

where the top horizontal map r is described in Definition 9.8, and the left-hand vertical map is determined by the inclusions

$$\text{Fun}_{\mathbb{N}}^{\simeq}((\mathcal{C}_0, \mathcal{W}_0), (\mathcal{C}_1, \mathcal{W}_1)) \subseteq \text{Fun}_{\mathbb{N}}((\mathcal{C}_0, \mathcal{W}_0), (\mathcal{C}_1, \mathcal{W}_1)).$$

Both maps are monomorphisms of ∞ -bicatogories, hence cofibrations in the scaled model structure on $\mathbf{Set}_\Delta^{\text{sc}}$, so the square is a homotopy pushout in that model structure. The scaled simplicial set $\mathbb{R}\text{el}_1\text{CAT}_\infty^{\text{diff}}$ is not itself an ∞ -bicategory, but we will eventually take a fibrant replacement of it in the scaled model structure to obtain an ∞ -bicategory $\mathbb{R}\text{elCAT}_\infty^{\text{diff}}$ on which to define the Goodwillie tangent structure.

Before that, however, we show that the scaled simplicial sets in (12.4) admit Weil-actions that are compatible with the inclusions, and hence determine a Weil-action on the pushout $\mathbb{R}\text{el}_1\text{CAT}_\infty^{\text{diff}}$.

DEFINITION 12.5. We define a map of simplicial sets

$$T : \text{Weil} \times \mathbb{R}\text{el}_0\text{CAT}_\infty^{\text{diff}} \rightarrow \mathbb{R}\text{el}_0\text{CAT}_\infty^{\text{diff}}$$

following a similar, but simpler, pattern to that in Definition 9.18. Recall from there that an n -simplex ϕ in Weil , with underlying labelled Weil-algebra morphisms

$$A_0 \xrightarrow{\phi_1} A_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_n} A_n,$$

determines a collection of functors

$$\tilde{\phi}_{i,j} : \mathcal{P}_{i,j} \rightarrow \text{Fun}(\mathcal{S}_{\text{fin},*}^{n_j}, \mathcal{S}_{\text{fin},*}^{n_i}).$$

An n -simplex λ in $\mathbb{R}\text{el}_0\text{CAT}_\infty^{\text{diff}}$ consists of a sequence of differentiable relative ∞ -categories

$$(\mathcal{C}_0, \mathcal{W}_0), \dots, (\mathcal{C}_n, \mathcal{W}_n)$$

and a collection of functors

$$\lambda_{i,j} : \mathcal{P}_{i,j} \rightarrow \text{Fun}(\mathcal{C}_i, \mathcal{C}_j)$$

which take values in the subcategories of differentiable relative functors

$$\text{Fun}_{\mathbb{N}}((\mathcal{C}_i, \mathcal{W}_i), (\mathcal{C}_j, \mathcal{W}_j)) \subseteq \text{Fun}(\mathcal{C}_i, \mathcal{C}_j).$$

We define $T^\phi(\lambda)$ to consist of the sequence

$$T^{A_0}(\mathcal{C}_0, \mathcal{W}_0), \dots, T^{A_n}(\mathcal{C}_n, \mathcal{W}_n)$$

together with the functors $T^\phi(\lambda)_{i,j}$ given by the composite

$$(12.6) \quad \begin{array}{c} \mathcal{P}_{i,j} \xrightarrow{\langle \tilde{\phi}_{i,j}, \lambda_{i,j} \rangle} \text{Fun}(\mathcal{S}_{\text{fin},*}^{n_j}, \mathcal{S}_{\text{fin},*}^{n_i}) \times \text{Fun}(\mathcal{C}_i, \mathcal{C}_j) \\ \xrightarrow{c} \text{Fun}(\text{Fun}(\mathcal{S}_{\text{fin},*}^{n_i}, \mathcal{C}_i), \text{Fun}(\mathcal{S}_{\text{fin},*}^{n_j}, \mathcal{C}_j)) \end{array}$$

where

$$c : \text{Fun}(\mathcal{S}_{\text{fin},*}^{n_j}, \mathcal{S}_{\text{fin},*}^{n_i}) \times \text{Fun}(\mathcal{C}_i, \mathcal{C}_j) \rightarrow \text{Fun}(\text{Fun}(\mathcal{S}_{\text{fin},*}^{n_i}, \mathcal{C}_i), \text{Fun}(\mathcal{S}_{\text{fin},*}^{n_j}, \mathcal{C}_j))$$

is adjoint to the composition map.

PROPOSITION 12.7. *The Weil-action map T of Definition 12.5 agrees with that of Definition 9.18 on the simplicial subset $\text{Weil} \times \mathbb{R}\text{el}_0\text{Cat}_\infty^{\text{diff}}$, and so determines an action*

$$T : \text{Weil} \times \mathbb{R}\text{el}_1\text{CAT}_\infty^{\text{diff}} \rightarrow \mathbb{R}\text{el}_1\text{CAT}_\infty^{\text{diff}}$$

of the scaled simplicial monoid Weil on the scaled simplicial set $\mathbb{R}\text{el}_1\text{CAT}_\infty^{\text{diff}}$.

PROOF. A similar argument to that of Lemma 9.19, but with a simpler argument for part (2) since the relevant edge is no longer a zigzag, implies that Definition 12.5 defines a scaled morphism

$$T : \text{Weil} \times \mathbb{R}\text{el}_0\text{CAT}_\infty^{\text{diff}} \rightarrow \mathbb{R}\text{el}_0\text{CAT}_\infty^{\text{diff}}.$$

A similar argument to that of Proposition 9.20 implies that T is an action of Weil on $\mathbb{R}\text{el}_0\text{CAT}_\infty^{\text{diff}}$.

The compatibility of Definitions 9.18 and 12.5 with respect to the inclusions $\text{Fun}(\mathcal{C}_i, \mathcal{C}_j) \subseteq \text{Ex}^\infty \text{Fun}(\mathcal{C}_i, \mathcal{C}_j)$ implies that the Weil -actions on $\mathbb{R}\text{el}_0\text{CAT}_\infty^{\text{diff}}$ and $\mathbb{R}\text{el}\text{Cat}_\infty^{\text{diff}}$ agree on their common simplicial subset $\mathbb{R}\text{el}_0\text{Cat}_\infty^{\text{diff}}$. It follows that these actions determine a Weil -action on the pushout $\mathbb{R}\text{el}_1\text{CAT}_\infty^{\text{diff}}$ as claimed. \square

Having established an action of Weil on $\mathbb{R}\text{el}_1\text{CAT}_\infty^{\text{diff}}$, we now take a fibrant replacement to obtain a corresponding action on an ∞ -bicategory $\mathbb{R}\text{el}\text{CAT}_\infty^{\text{diff}}$. For that purpose we introduce the following model structure on the category of scaled simplicial sets with an action of Weil .

PROPOSITION 12.8. *Let $\text{Mod}_{\text{Weil}}^{\text{sc}}$ be the category of scaled Weil -modules, i.e. the category of modules over Weil , viewed as a monoid in $\text{Set}_\Delta^{\text{sc}}$ with its maximal scaling. Then $\text{Mod}_{\text{Weil}}^{\text{sc}}$ has a model structure in which a morphism is a weak equivalence (or fibration) if and only if the underlying map of scaled simplicial sets is a weak equivalence (or fibration).*

PROOF. We apply Schwede-Shipley's result [SS00, 4.1] to the scaled simplicial monoid Weil . By [SS00, 4.2] we must verify that the scaled model structure on $\text{Set}_\Delta^{\text{sc}}$ is monoidal with respect to the cartesian product, which is done in [Dev16, 2.1.21]. \square

DEFINITION 12.9. Let $\mathbb{R}\text{el}\text{CAT}_\infty^{\text{diff}}$ be the scaled Weil -module given by a fibrant replacement, in the model structure of Proposition 12.8, of the Weil -action on $\mathbb{R}\text{el}_1\text{CAT}_\infty^{\text{diff}}$ described in Proposition 12.7.

The scaled simplicial set $\mathbb{R}\text{el}\text{CAT}_\infty^{\text{diff}}$ is an ∞ -bicategory, and we can choose the fibrant replacement so that the comparison map

$$\mathbb{R}\text{el}_1\text{CAT}_\infty^{\text{diff}} \xrightarrow{\sim} \mathbb{R}\text{el}\text{CAT}_\infty^{\text{diff}}$$

is a cofibration, hence a monomorphism, of scaled simplicial sets. Altogether we have produced the following diagram of inclusions of ∞ -bicategories, with compatible Weil -actions, which is also a homotopy pushout of $(\infty, 2)$ -categories:

$$(12.10) \quad \begin{array}{ccc} \mathbb{R}\text{el}_0\text{Cat}_\infty^{\text{diff}} & \xrightarrow{r} & \mathbb{R}\text{el}\text{Cat}_\infty^{\text{diff}} \\ \downarrow & & \downarrow \\ \mathbb{R}\text{el}_0\text{CAT}_\infty^{\text{diff}} & \xrightarrow{\quad} & \mathbb{R}\text{el}\text{CAT}_\infty^{\text{diff}} \end{array}$$

We now show that $\mathbb{R}\text{el}\text{CAT}_\infty^{\text{diff}}$ is a model for the $(\infty, 2)$ -category $\text{CAT}_\infty^{\text{diff}}$ of differentiable ∞ -categories described in Definition 12.1.

PROPOSITION 12.11. *The composite map*

$$M : \text{CAT}_\infty^{\text{diff}} \xrightarrow{M_0} \mathbb{R}\text{el}_0\text{CAT}_\infty^{\text{diff}} \xrightarrow{\quad} \mathbb{R}\text{el}\text{CAT}_\infty^{\text{diff}}$$

is an equivalence of ∞ -bicategories, where $M_0 : \mathbb{C}AT_\infty^{\text{diff}} \rightarrow \mathbb{R}el_0\mathbb{C}AT_\infty^{\text{diff}}$ is the $\mathbb{C}at_\infty$ -enriched functor given by $\mathcal{C} \mapsto (\mathcal{C}, \mathcal{E}_{\mathcal{C}})$.

PROOF. Our strategy is to model the inclusion $\mathbb{R}el_0\mathbb{C}AT_\infty^{\text{diff}} \rightarrow \mathbb{R}el\mathbb{C}AT_\infty^{\text{diff}}$ in the context of marked simplicial categories by translating the homotopy pushout (12.10) back into that world. Consider the diagram of marked simplicial categories

$$(12.12) \quad \begin{array}{ccc} (\mathbb{R}el_0\mathbb{C}at_\infty^{\text{diff}}, \simeq) & \longrightarrow & (\mathbb{R}el_0\mathbb{C}at_\infty^{\text{diff}}, \mathbb{R}el_0\mathbb{C}at_\infty^{\text{diff}}) \\ \downarrow & & \downarrow \\ (\mathbb{R}el_0\mathbb{C}AT_\infty^{\text{diff}}, \simeq) & \longrightarrow & (\mathbb{R}el_0\mathbb{C}AT_\infty^{\text{diff}}, \mathbb{R}el_0\mathbb{C}at_\infty^{\text{diff}}) \end{array}$$

where a pair (\mathbb{C}, \mathbb{M}) denotes the simplicial category \mathbb{C} with markings given by those edges in the subcategory \mathbb{M} . We use the notation \simeq to denote the natural marking of a $\mathbb{C}at_\infty$ -category: the subcategory consisting of all the equivalences in the mapping objects. The horizontal functors in (12.12) are given by the identity map, and the vertical functors by the inclusion $\mathbb{R}el_0\mathbb{C}at_\infty^{\text{diff}} \subseteq \mathbb{R}el_0\mathbb{C}AT_\infty^{\text{diff}}$.

We claim that (12.12) is a homotopy pushout diagram in the model structure on marked simplicial categories described by Lurie in [Lur09a, A.3.2]. The top horizontal map is a cofibration because it has the left lifting property with respect to acyclic fibrations, and the square is a strict pushout of marked simplicial categories. Since the model structure on marked simplicial categories is left proper by [Lur09a, A.3.2.4], it follows that (12.12) is a homotopy pushout in that model structure.

Consider the top-right corner of (12.12): there is a marked simplicial functor of maximally marked simplicial categories

$$(\mathbb{R}el_0\mathbb{C}at_\infty^{\text{diff}}, \mathbb{R}el_0\mathbb{C}at_\infty^{\text{diff}}) \xrightarrow{r} (\mathbb{R}el\mathbb{C}at_\infty^{\text{diff}}, \mathbb{R}el\mathbb{C}at_\infty^{\text{diff}}) = (\mathbb{R}el\mathbb{C}at_\infty^{\text{diff}}, \simeq)$$

given on mapping objects by the map r of Definition 9.8. To see that this functor is an equivalence of marked simplicial categories, we note that the maximal marking functor takes an acyclic cofibration of simplicial sets (in the Quillen model structure), such as each r_Y , to an equivalence in the marked model structure. This fact can be checked directly from the definition of marked (cartesian) equivalence in [Lur09a, 3.1.3.3].

We have now done enough to establish that the homotopy pushout square (12.12) corresponds, under the Quillen equivalence of [Lur09b, 4.2.7], to the homotopy pushout square (12.10), and hence that the map $\mathbb{R}el_0\mathbb{C}AT_\infty^{\text{diff}} \rightarrow \mathbb{R}el\mathbb{C}AT_\infty^{\text{diff}}$ can be modelled by the marked simplicial functor

$$(\mathbb{R}el_0\mathbb{C}AT_\infty^{\text{diff}}, \simeq) \rightarrow (\mathbb{R}el_0\mathbb{C}AT_\infty^{\text{diff}}, \mathbb{R}el_0\mathbb{C}at_\infty^{\text{diff}})$$

that is the identity on the underlying simplicial category. The desired proposition is therefore reduced to showing that the functor

$$M : (\mathbb{C}AT_\infty^{\text{diff}}, \simeq) \rightarrow (\mathbb{R}el_0\mathbb{C}AT_\infty^{\text{diff}}, \mathbb{R}el_0\mathbb{C}at_\infty^{\text{diff}}); \quad \mathcal{C} \mapsto (\mathcal{C}, \mathcal{E}_{\mathcal{C}})$$

is an equivalence of marked simplicial categories. Given two differentiable ∞ -categories $\mathcal{C}_0, \mathcal{C}_1$, we have

$$\text{Fun}_{\mathbb{N}}(\mathcal{C}_0, \mathcal{C}_1) = \text{Fun}_{\mathbb{N}}((\mathcal{C}_0, \mathcal{E}_{\mathcal{C}_0}), (\mathcal{C}_1, \mathcal{E}_{\mathcal{C}_1}))$$

so M is fully faithful. The proof that M is essentially surjective on objects follows by the construction in the proof of Proposition 9.7 with no changes. \square

We now transfer the Weil-action on the ∞ -bicategory $\mathbb{R}\text{elCAT}_\infty^{\text{diff}}$ along the equivalence $M : \text{CAT}_\infty^{\text{diff}} \xrightarrow{\sim} \mathbb{R}\text{elCAT}_\infty^{\text{diff}}$ of Proposition 12.11. This transfer requires an ∞ -bicategorical version of Lemma 3.12.

LEMMA 12.13. *Let $i : \mathbb{X} \xrightarrow{\sim} \mathbb{Y}$ be an equivalence of ∞ -bicategories. Then there is an equivalence of monoidal ∞ -categories*

$$\text{End}_{(\infty, 2)}(\mathbb{X}) \simeq \text{End}_{(\infty, 2)}(\mathbb{Y})$$

whose underlying functor is equivalent to $i(-)i^{-1}$.

PROOF. The method of proof for Lemma 3.12 applies in exactly the same way, using the fact that the construction $\text{Fun}_{(\infty, 2)}(-, -) \simeq$ takes an equivalence of ∞ -bicategories (in either of its variables) to an equivalence of ∞ -categories. \square

DEFINITION 12.14. Let

$$T : \text{Weil}^\otimes \rightarrow \text{End}_{(\infty, 2)}(\text{CAT}_\infty^{\text{diff}})^\circ$$

be the monoidal functor obtained by composing the action map

$$\text{Weil}^\otimes \rightarrow \text{End}_{(\infty, 2)}(\mathbb{R}\text{elCAT}_\infty^{\text{diff}})^\circ$$

associated to Definition 12.9 with the equivalence of monoidal ∞ -categories induced, via Lemma 12.13, by the equivalence $\text{CAT}_\infty^{\text{diff}} \xrightarrow{\sim} \mathbb{R}\text{elCAT}_\infty^{\text{diff}}$ of Proposition 12.11.

THEOREM 12.15. *The map T of Definition 12.14 is a tangent structure on the ∞ -bicategory $\text{CAT}_\infty^{\text{diff}}$ which (up to equivalence) extends that of Theorem 9.22 on the ∞ -category $\text{Cat}_\infty^{\text{diff}}$.*

PROOF. To show that T is a tangent structure, we apply Proposition 6.20. The only thing remaining to show is that the foundational and vertical lift pullbacks in Weil determine homotopy 2-pullbacks in $\text{CAT}_\infty^{\text{diff}}$ for each object $\mathcal{C} \in \text{CAT}_\infty^{\text{diff}}$. In the proof of Theorem 9.22 we showed each such square is a pullback along a fibration in the ∞ -cosmos $\text{CAT}_\infty^{\mathbb{N}}$, which immediately implies that claim. \square

Goodwillie calculus in an ∞ -bicategory. In this final section of the paper we show how to use Theorem 11.3 to define a notion of P_n -equivalence, and hence Taylor tower, in an arbitrary tangent ∞ -bicategory \mathbb{X} which, when $\mathbb{X} = \text{CAT}_\infty^{\text{diff}}$, recovers Goodwillie’s theory.

DEFINITION 12.16. Let (\mathbb{X}, T) be a tangent ∞ -bicategory that admits a terminal object $*$, and let $x : * \rightarrow \mathcal{C}$ be a 1-morphism in \mathbb{X} , i.e. a generalized object in \mathcal{C} . We say that \mathcal{C} admits higher tangent spaces at x if, for each n , there is a homotopy 2-pullback in \mathbb{X} of the form

$$\begin{array}{ccc} T_x^n \mathcal{C} & \xrightarrow{\iota_x^n} & T^n(\mathcal{C}) \\ \downarrow & & \downarrow p^n \\ * & \xrightarrow{x} & \mathcal{C} \end{array}$$

Here p^n denotes the natural transformation associated to the labelled Weil-algebra morphism given by the augmentation $W^{\otimes n} \rightarrow \mathbb{N}$.

EXAMPLE 12.17. When $\mathbb{X} = \mathbb{C}\text{AT}_{\infty}^{\text{diff}}$, a morphism $x : * \rightarrow \mathcal{C}$ as in Definition 12.16 is an actual object of a differentiable ∞ -category \mathcal{C} which admits the higher tangent spaces $T_x^n \mathcal{C}$ as described in Theorem 11.3.

DEFINITION 12.18. Let (\mathbb{X}, T) be a tangent ∞ -bicategory, and suppose the object \mathcal{C} in \mathbb{X} admits higher tangent spaces at x . For any $\mathcal{D} \in \mathbb{X}$ and $n \geq 0$, we define the *subcategory of P_n^x -equivalences*

$$\mathcal{P}_n^x \text{Hom}_{\mathbb{X}}(\mathcal{C}, \mathcal{D}) \subseteq \text{Hom}_{\mathbb{X}}(\mathcal{C}, \mathcal{D})$$

to be the subcategory of morphisms that map to equivalences under the functor

$$T^n \iota_x^n : \text{Hom}_{\mathbb{X}}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Hom}_{\mathbb{X}}(T_x^n \mathcal{C}, T_x^n \mathcal{D}); \quad F \mapsto T^n(F) \iota_x^n.$$

LEMMA 12.19. *Let (\mathbb{X}, T) and $x : * \rightarrow \mathcal{C}$ be as in Definition 12.18. Then for all $n \geq 1$:*

$$\mathcal{P}_n^x \text{Hom}_{\mathbb{X}}(\mathcal{C}, \mathcal{D}) \subseteq \mathcal{P}_{n-1}^x \text{Hom}_{\mathbb{X}}(\mathcal{C}, \mathcal{D}).$$

PROOF. It is sufficient to show that, up to natural equivalence, we have a factorization

$$(12.20) \quad \begin{array}{ccc} \text{Hom}_{\mathbb{X}}(\mathcal{C}, \mathcal{D}) & \xrightarrow{T^n \iota_x^n} & \text{Hom}_{\mathbb{X}}(T_x^n \mathcal{C}, T_x^n \mathcal{D}) \\ & \searrow^{T^{n-1}, \iota_x^{n-1}} & \downarrow \\ & & \text{Hom}_{\mathbb{X}}(T_x^{n-1} \mathcal{C}, T_x^{n-1} \mathcal{D}) \end{array}$$

where the vertical map is given by postcomposition with $p_{T^{n-1} \mathcal{D}} : T_x^n \mathcal{D} \rightarrow T_x^{n-1} \mathcal{D}$ and precomposition with the map $0_x^n : T_x^{n-1} \mathcal{C} \rightarrow T_x^n \mathcal{C}$ induced by the bottom homotopy 2-pullback square in the following diagram

$$\begin{array}{ccccc} T_x^{n-1} \mathcal{C} & \longrightarrow & T_x^{n-1} \mathcal{C} & & \\ & \searrow^{0_x^n} & & \searrow^{0_{T_x^{n-1} \mathcal{C}}} & \\ & & T_x^n \mathcal{C} & \longrightarrow & T_x^n \mathcal{C} \\ & & \downarrow & & \downarrow p_{\mathcal{C}}^n \\ & & \mathcal{A} & \xrightarrow{x} & \mathcal{C} \end{array}$$

Note that $p_{\mathcal{C}}^{n-1} \simeq p_{\mathcal{C}}^n 0_{T_x^{n-1} \mathcal{C}}$ by uniqueness of the augmentation map $W^{\otimes(n-1)} \rightarrow \mathbb{N}$. Finally, the diagram (12.20) commutes since

$$p_{T_x^{n-1} \mathcal{D}} T^n F 0_{T_x^{n-1} \mathcal{C}} \simeq T^{n-1} F$$

by that same calculation together with the naturality of T . \square

EXAMPLE 12.21. Let $\mathbb{X} = \mathbb{C}\text{AT}_{\infty}^{\text{diff}}$, and let x be an object in a differentiable ∞ -category \mathcal{C} which admits finite colimits. Then Theorem 11.3 tells us that

$$\mathcal{P}_n^x \text{Hom}_{\mathbb{C}\text{AT}_{\infty}^{\text{diff}}}(\mathcal{C}, \mathcal{D}) \subseteq \text{Fun}_{\mathbb{N}}(\mathcal{C}, \mathcal{D})$$

is precisely the subcategory of P_n^x -equivalences between functors $\mathcal{C} \rightarrow \mathcal{D}$ which preserve sequential colimits.

We recover Goodwillie's notion of n -excisive functor $\mathcal{C} \rightarrow \mathcal{D}$, and hence the Taylor tower, by observing that in $\mathbb{C}\text{AT}_{\infty}^{\text{diff}}$ the subcategory of P_n^x -equivalences is

associated with a left exact localization of $\mathrm{Fun}_{\mathbb{N}}(\mathcal{C}, \mathcal{D})$. The local objects for that localization are the n -excisive functors. We generalize this observation to give a definition of Taylor tower in an arbitrary tangent ∞ -bicategory.

DEFINITION 12.22. Let \mathbb{X} be a tangent ∞ -bicategory, and suppose that an object \mathcal{C} in \mathbb{X} admits higher tangent spaces at a generalized object $x : * \rightarrow \mathcal{C}$. We say that \mathbb{X} *admits Taylor towers expanded at x* if, for each $\mathcal{D} \in \mathbb{X}$ and each $n \geq 0$, there is a full subcategory

$$\mathrm{Jet}_x^n(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Hom}_{\mathbb{X}}(\mathcal{C}, \mathcal{D}),$$

for which the inclusion admits a left adjoint P_n^x such that the subcategory of P_n^x -equivalences, in the sense of 12.18, is equal to the subcategory of morphisms that are mapped to equivalences by P_n^x . We can refer to $\mathrm{Jet}_x^n(\mathcal{C}, \mathcal{D})$ as *the ∞ -category of n -jets at x* for morphisms $\mathcal{C} \rightarrow \mathcal{D}$ in \mathbb{X} .

In that case, by Lemma 12.19, we necessarily have

$$\mathrm{Jet}_x^{n-1}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Jet}_x^n(\mathcal{C}, \mathcal{D})$$

and for each 1-morphism $F : \mathcal{C} \rightarrow \mathcal{D}$ in \mathbb{X} , there is a sequence of morphisms in $\mathrm{Hom}_{\mathbb{X}}(\mathcal{C}, \mathcal{D})$ of the form

$$F \rightarrow \cdots \rightarrow P_n^x F \rightarrow P_{n-1}^x F \rightarrow \cdots \rightarrow P_0^x F$$

which we can call the *Taylor tower* of F at x . Taking \mathbb{X} to be the Goodwillie tangent structure on the ∞ -bicategory $\mathrm{CAT}_{\infty}^{\mathrm{diff}}$, we recover Goodwillie's notion of Taylor tower for functors between differentiable ∞ -categories.

Proposals for Future Work

The work in this paper is intended to open up various avenues for further research, and we conclude by describing ideas for projects that build on the concepts developed here. Some of those ideas were mentioned in the introduction (under [Connections and Conjectures](#)), and we now expand a little on some of those suggestions.

Vector bundles in Goodwillie calculus. A central role in differential geometry is of course played by vector bundles. The corresponding notion in an abstract tangent category is a *differential bundle* introduced by Cockett and Cruttwell and explored in detail in [\[CC18\]](#). MacAdam [\[Mac21\]](#) has proved that this abstract definition recovers the standard notion of vector bundle in the tangent category of smooth manifolds.

We proved in Chapter [10](#) that the analogues of vector *spaces* in Goodwillie calculus are the stable ∞ -categories. It is clear therefore that a differential bundle in the tangent ∞ -category $\text{Cat}_\infty^{\text{diff}}$ should consist of a functor

$$q : \mathcal{E} \rightarrow \mathcal{M}$$

for which the fibre \mathcal{E}_X over any object $X \in \mathcal{M}$ is a stable ∞ -category. However, it is less clear what additional conditions the functor q should be required to satisfy.

One challenge here is that Cockett and Cruttwell’s original definition of differential bundle [\[CC18, Def. 2.3\]](#) does not easily extend to the ∞ -categorical case. In order to describe the differential bundles in $\text{Cat}_\infty^{\text{diff}}$ we first need a characterization in terms of Weil-algebras akin to the description of differential objects given in [Proposition 5.8](#).

Some guidance to finding that characterization might be given by the approach in [\[CC18, Sec. 3\]](#), in which differential bundles can in some cases be identified with differential objects in a *slice* tangent ∞ -category, or in the work of MacAdam [\[Mac21, 2.2\]](#), in which there is an alternative presentation of differential bundles which could be more amenable to translation into the ∞ -categorical setting.

Connections, curvature, etc. In the context of abstract tangent categories, notions of connection were introduced by Cockett and Cruttwell in [\[CC17\]](#) and further developed by Lucyshyn-Wright in [\[Luc17\]](#). Roughly speaking, a *connection* on an object M in a tangent category \mathbb{X} consists of a morphism

$$K : T^2M \rightarrow TM$$

such that the triple

$$\langle Tp, K, pT \rangle : T^2M \rightarrow TM \times_M TM \times_M TM$$

is an isomorphism (additional linearity conditions are also required). This decomposition of the double tangent bundle allows for constructions such as parallel transport [CC17, 5.20] to be realized in an abstract tangent category.

In the tangent ∞ -category $\text{Cat}_{\infty}^{\text{diff}}$ there is a natural candidate for such a connection on any differentiable ∞ -category \mathcal{C} ; the functor

$$K : T^2\mathcal{C} \rightarrow T\mathcal{C}$$

is given by a form of multilinearization. That construction appears to be a connection under some limited circumstances, for example if \mathcal{C} is a stable ∞ -category, but not in general. Nonetheless, K does appear to be a *vertical* connection in the sense of [CC17, 3.2], and any differentiable ∞ -category \mathcal{C} seems to also admit a *horizontal connection* H , though typically H and K are not exactly compatible in the sense described in [CC17, 5.2].

Curvature is defined in [CC17, 3.17] for an object in a tangent category equipped with a (vertical) connection. With that definition, the vertical connection K described above appears to always be ‘flat’ (i.e. has zero curvature), so perhaps this version of curvature is not the right concept to focus on in the context of functor calculus.

As mentioned in the introduction there are many other concepts from differential geometry that have been translated into the abstract setting, including affine spaces, differential forms, and Lie algebroids. We do not have specific ideas about how these concepts might manifest in the Goodwillie tangent structure, but each would be worthy of study in order to understand how they might be related to ideas from homotopy theory.

Other brands of functor calculus. This paper is concerned with what is sometimes referred to as Goodwillie’s ‘homotopy’ calculus, but there are other versions of functor calculus we could consider.

Goodwillie and Weiss [Wei99, GW99] developed a ‘manifold’ calculus and used it to investigate spaces of embeddings. That theory focuses on presheaves (with values in some ∞ -category) on a fixed smooth manifold M , and describes how global sections can be recovered from higher-order local information.¹ As in the homotopy calculus, there are Taylor towers whose terms play the role of ‘polynomial’ approximations, and under suitable circumstances that tower ‘converges’ to the global sections of the presheaf of interest.

We are curious if there is a tangent ∞ -category that encodes the Goodwillie-Weiss manifold calculus in the same way the Goodwillie tangent structure of this paper describes homotopy calculus. For example, the tangent bundle on an ∞ -category \mathcal{C} might be given by the ∞ -category of presheaves on M , with values in \mathcal{C} , that are *degree* ≤ 1 in the sense described in [Wei99, 2.2]. Preliminary calculations suggest that this construction of a tangent bundle does *not* satisfy the full vertical lift axiom (see 8.36) for a tangent category, though perhaps a weaker notion could still apply. A similar approach could be taken with the ‘orthogonal’

¹The premise of this paper is that Goodwillie’s homotopy calculus admits a close analogy with the theory of smooth manifolds. The manifold calculus, on the other hand, actually concerns manifolds themselves, not via analogy. The imaginative reader might therefore consider how to generalize the manifold calculus to an abstract tangent ∞ -category, and hence develop a ‘Goodwillie calculus’ calculus, in which ideas from Goodwillie and Weiss are used to study presheaves on a site associated to a fixed differentiable ∞ -category.

calculus of Weiss [Wei95], in which the objects of interests are functors on a certain ∞ -category of real inner product spaces, or the related ‘unitary’ calculus of Taggart [Tag22].

The homotopy calculus itself also has an equivariant version developed by Dotto [Dot16, Dot17] based on ideas of Blumberg [Blu06]. In this context, excisive functors are classified by genuine equivariant spectra. Perhaps there is a suitable tangent ∞ -category based on these constructions which bears a closer connection to modern equivariant homotopy theory than the version developed in this paper.

Goodwillie tangent structure on an ∞ -cosmos. In a series of papers starting with [RV17] Riehl and Verity have developed the notion of an ∞ -cosmos which captures some of the features of the collection of ∞ -categories and is intended as a model-independent context for ∞ -category theory. We used that work in the proof of Theorem 9.22 by considering the ∞ -cosmos of ∞ -categories that admit sequential colimits.

It seems reasonable to expect that the existence of the Goodwillie tangent structure could be extended to a wider collection of ∞ -cosmoses. In particular, any ∞ -cosmos \mathbb{K} has cotensors by quasicategories, so that it makes sense to consider the object $\text{Fun}(\mathcal{S}_{\text{fin},*}, \mathcal{C})$ for an object \mathcal{C} in \mathbb{K} , and appropriate limits so that an analogue of the tangent bundle $T\mathcal{C} = \text{Exc}(\mathcal{S}_{\text{fin},*}, \mathcal{C})$ can also be constructed. One might hope to identify the conditions on the ∞ -cosmos \mathbb{K} in which that tangent bundle underlies a tangent structure in the sense of this paper.

Operads and tangent $(\infty, 2)$ -categories. In many of the proposals above, we have suggested how to apply the theory of abstract tangent categories to Goodwillie calculus and homotopy theory, but we can also look for applications in the reverse direction, by seeing if concepts from Goodwillie calculus can be generalized to the abstract setting.

Greg Arone and the third author examined in [AC11] the role of operads in Goodwillie calculus, formulating a chain rule for derivatives in that context. It is reasonable to ask whether these operad structures reflect a more general phenomenon. We have seen that the natural setting for Goodwillie calculus is a tangent $(\infty, 2)$ -category, and that might be the appropriate place to identify such a generalization. Clues to the right approach here may be in work of Lemay [Lem18], which relates the Faà di Bruno formula (the chain rule for higher-order derivatives in ordinary calculus) to tangent categories, building on work of Cockett and Seeley [CS11].

Higher approximations to ∞ -categories. Heuts [Heu21, 1.7] has introduced a theory of Goodwillie towers for pointed compactly-generated ∞ -categories, instead of functors. That construction provides each such ∞ -category \mathcal{C} with a sequence of approximations $\mathcal{P}_n\mathcal{C}$. The first approximation $\mathcal{P}_1\mathcal{C}$ is the stabilization $Sp(\mathcal{C})$, i.e. equal to the tangent space $T_*\mathcal{C}$ to \mathcal{C} at the null object. It would be interesting to see if these higher approximations also admit a description in terms of the Goodwillie tangent structure, presumably somehow related to the theory of n -jets in Chapter 11. Alternatively, one might find a connection between Heuts’s theory and ‘higher’ tangent structures based on Weil-algebras whose generating relations are higher degree monomials. For example, do Heuts’s n -excisive approximations correspond to Weil-algebras of the form $\mathbb{N}[x]/(x^{n+1})$?

Cartesian differential ∞ -categories. In Chapter 5 we showed that a certain homotopy category of differential objects in a cartesian tangent ∞ -category form a cartesian differential category in the sense of Blute, Cockett and Seeley [BCS09]. We are curious whether there is a sensible notion of cartesian differential ∞ -category which refines that construction (as well as the corresponding construction of the first author and others [BJO⁺18] in the context of abelian categories) without the need to take a homotopy category. As usual, the original definition of cartesian differential category does not easily generalize to the ∞ -category context. Based on Theorem 10.2 we would expect the ∞ -category $\mathbb{C}at_{\infty}^{\text{diff, st}}$ to be an example of a cartesian differential ∞ -category, but we do not have a precise definition of such a structure.

Derived manifolds and synthetic differential geometry. This paper has been primarily focused on the tangent structure that encodes functor calculus, but we did provide another example of a tangent ∞ -category in Proposition 3.15: the ∞ -category of derived manifolds. We have not explored any features of that tangent structure in this paper, so much remains to be done. For example, what are the differential objects or differential bundles for derived manifolds, and how do they relate to ordinary vector spaces and vector bundles? We might also hope for there to be a close connection between derived manifolds and an ∞ -categorical version of synthetic differential geometry [Koc81].

Tangent structures on ring spectra and spectral algebraic geometry. In Example 1.14(3), we mentioned a fundamental example of a tangent structure on the category $\mathbb{C}Ring$ of commutative rings with tangent bundle functor $T(R) = R[x]/(x^2)$. It follows from [CC14, 5.17] that there is a corresponding ‘dual’ tangent structure on $\mathbb{C}Ring^{op}$ with tangent bundle given by (the opposite of) the left adjoint U to T . That left adjoint can be described in terms of Kähler differentials; see [CC14, 5.16]. Moreover, this tangent structure on $\mathbb{C}Ring^{op}$ extends to the tangent category of schemes mentioned in Example 1.14(2).

Given that one of the themes of this paper is the extension of ideas from ordinary tangent categories to ∞ -categories, it is natural to ask whether the examples mentioned above correspond to tangent structures on the ∞ -category $\mathbb{C}Ring_{\infty}$ of commutative (or E_{∞}) ring *spectra* and its opposite. Such structures may be the basis for tangent ∞ -categories that play a role in spectral algebraic geometry, e.g. for Lurie’s spectral Deligne-Mumford stacks [Lur18, 1.4.4.2].

These questions also raise the possibility of a different, perhaps more sophisticated, notion of tangent ∞ -category than the one developed in this paper. Recall that our definition of tangent ∞ -category is based on ‘ordinary’ Weil-algebras, i.e. certain commutative semi-rings and their homomorphisms. What if instead we started with an ∞ -category whose objects are ‘semi-rings up to homotopy’, perhaps a suitable ∞ -category of E_{∞} -ring spaces? Could one then develop a theory of tangent ∞ -categories that adds higher structure to the concepts studied in this paper, and which incorporates more subtle aspects of homotopy theory? We do not yet have any suggestions for what such a theory could reveal.

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