CALCULUS OF FUNCTORS AND CONFIGURATION SPACES

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Introduction

This is a summary of a talk given at the Conference on Pure and Applied Topology on the Isle of Skye from June 21-25, 2005. The author would like to thank the organisers of the conference for a fantastic week and for the opportunity to present the following work.

We describe a relationship between Goodwillie's calculus of homotopy functors and configuration spaces. In [3], we showed that the Goodwillie derivatives of the identity functor on based spaces form an operad of spectra. Here we show that the configuration spaces of points in a parallelizable manifold form, up to suspension and homotopy, a right module over this operad (see Proposition 3.1). We then describe how this construction might be related to work of Markl, in which he shows that these configuration spaces form a right module over the Fulton-MacPherson operads F_m constructed from the compactified configuration spaces of points in $\mathbb{R}^m$.

We refer the reader to Goodwillie [5] for background on the calculus of homotopy functors, and to Kathryn Hess's talk at this same conference for background on operads and modules over them. In §1 below, we summarize the construction of an operad structure on the derivatives of the identity functor. In §2, we construct from a fixed based space X, a right module C_X over that operad. In §3, we use Atiyah duality to relate this module to configuration spaces for parallelizable manifolds. Finally, in §4, we recall the work of Markl, and conjecture a connection between his constructions and ours.

Much of this work is still in progress, and we warn the reader that not all the details have been fully worked out.

1. Derivatives of the identity

The identity functor on based spaces, which we write

$$I : \mathsf{Top}_* \to \mathsf{Top}_*,$$

is a fundamental example in the calculus of homotopy functors. We write $\partial_n I$ for the n^{th} derivative of I. These derivatives were calculated by Johnson [6] and Arone-Mahowald [2].

Proposition 1.1 (Arone-Mahowald)**.**

$$\partial_n I \simeq \mathrm{Map}_*(K_n, \mathbb{S}),$$

where K_n *denotes the* partition poset complex *and* $\mathbb{S}$ *is the sphere spectrum. The partition poset complex, defined below, is a finite cell complex with action of the symmetric group* Σ_n*.*

Definition 1.2. Let $n \geq 1$ be a fixed integer and write $K(n)$ for the poset of partitions of the set $\{1, \dots, n\}$, ordered by refinement. We denote by $\hat{0}$ the minimum element in this poset, the partition consisting of singletons, and by $\hat{1}$ the maximum element, the partition with only one piece. Let $K_0(n) := K(n) - \hat{0}$ and $K_1(n) := K(n) - \hat{1}$.

The *partition poset complex* K_n is the following geometric realization of a simplicial set constructed from the nerves of the categories $K(n)$, $K_0(n)$ and $K_1(n)$:

$$K_n := \left| \frac{N_\bullet K(n)}{N_\bullet K_0(n) \cup N_\bullet K_1(n)} \right|.$$

The symmetric group Σ_n acts on $K(n)$ by permuting the elements of the set $\{1, \dots, n\}$ and this induces an action on K_n.

The connection between the partition poset complexes and the theory of operads comes via the bar construction which we now define.

Definition 1.3. Let P be an augmented operad in the symmetric monoidal category $\mathcal{C}$. The *reduced simplicial bar construction on* P is the simplicial symmetric sequence $B_\bullet(P)$ given by

$$B_n(P) := \underbrace{P \circ \dots \circ P}_{n}$$

with face maps given by the operad composition $P \circ P \to P$ and the augmentation $P \to 1$, and degeneracy maps given by the operad unit map $1 \to P$. (Here, $\circ$ is the composition product of symmetric sequences and 1 denotes the unit symmetric sequence in $\mathcal{C}$.) If $\mathcal{C}$ is enriched and tensored over topological spaces, we write $B(P)$ for the geometric realization of $B_\bullet(P)$. We call this the *reduced bar construction on* P. It is a symmetric sequence in the category $\mathcal{C}$.

The partition poset complexes are a special case of this construction.

Example 1.4. Let $\underline{S^0}$ be the operad in Top_* (with symmetric monoidal structure given by smash product) with $\underline{S^0}(n) := S^0$ for all n with the obvious isomorphisms for the operad compositions. Then the partition poset complexes are homeomorphic to the pieces of the bar construction on $\underline{S^0}$:

$$K_n \cong B(\underline{S^0})(n).$$

The following result of [3] was also proved by Salvatore [8].

Proposition 1.5. *Let P be an operad in Top_* with $P(1) \cong S^0$. Then $B(P)$ has a natural cooperad structure.*

Corollary 1.6. *The partition poset complexes form a cooperad and, dually, the Goodwillie derivatives of the identity form an operad.*

Proof. Taking the Spanier-Whitehead duals of the cooperad structure maps on the partition poset complexes yields operad structure maps for the models of the $\partial_n I$ of Proposition 1.1. □

2. Modules over the derivatives of the identity

Given a right, respectively left, module M over the operad P there is a bar construction with coefficients that produces a right, respectively left, comodule $B(M)$ over the cooperad $B(P)$. We use this construction to get modules over the operad $\partial_* I$.

Let X be a based topological space. We define a right $\underline{S^0}$-module $X^\wedge$ by

$$X^\wedge(n) := X^{\wedge n},$$

the n-fold smash product of X. The module structure maps take the form

$$X^{\wedge k} \wedge S^0 \wedge \ldots \wedge S^0 \to X^{\wedge n_1 + \cdots + n_k}$$

and are given by the reduced diagonal $X \to X^{\wedge n_i}$ on each of the k factors of X in the source.

Applying the bar construction to this $\underline{S^0}$-module, we get a right comodule over the partition poset complexes:

$$M_X := B(X^\wedge)$$

and, by taking the Spanier-Whitehead dual, a right module over the derivatives of the identity:

$$C_X := \mathrm{Map}_*(M_X, \mathbb{S}).$$

This construction determines a *contravariant* functor from Top_* to the category of right $\partial_* I$-modules.

The explicit definition of the bar construction in terms of trees allows us to identify the terms in the symmetric sequence M_X and hence C_X:

Lemma 2.1. *Let X be a based CW complex. With C_X defined as above, we have*

$$C_X(n) \simeq \mathrm{Map}_*(X^{\wedge n}/\Delta, \mathbb{S})$$

where Δ denotes the 'fat diagonal' in $X^{\wedge n}$, that is the subspace

$$\Delta := \{(x_1, \dots, x_n) \in X^{\wedge n} \mid x_i = x_j \text{ for some } i \neq j\}.$$

Remark 2.2. Let K be a based finite CW complex. Arone has shown in [1] that the n^{th} derivative of the functor $X \mapsto \Sigma^\infty \mathrm{Map}_*(K, X)$ is the spectrum

$$\mathrm{Map}_*(K^{\wedge n}/\Delta, \mathbb{S}).$$

By Lemma 2.1, this is equivalent to $C_K(n)$. The collection of derivatives of this functor from based spaces to spectra thus form a right module over the derivatives of the identity.

3. Configuration spaces for parallelizable modules

In this section we analyze the module C_X when X is the one-point compactification of a parallelizable manifold M. We show that, in this case, C_X is, up to suspensions and homotopy, composed of the configuration spaces of points in the manifold M. This is just a simple consequence of Lemma 2.1 using Atiyah duality.

Let M be a parallelizable manifold and write M^+ for its one-point compactification. If M is already compact then M^+ stands for M with a disjoint basepoint. Since M^+ is a based space we can apply the constructions of §2. The result is the following proposition.

Proposition 3.1. *Let M be a parallelizable m-dimensional manifold with one-point compactification M^+. Then*

$$C_{M^+}(n) \simeq \Sigma^{-mn}\Sigma^\infty C(M,n)_+$$

where $C(M,n)_+$ denotes the configuration space of n distinct points in M, together with a disjoint basepoint. These stable configuration spaces, therefore, form a right module over the derivatives of the identity functor.

Sketch proof. The key point of this proof is that $C(M,n)$ is equal to M^n minus the fat diagonal. Since M is parallelizable, this is Atiyah dual to M^n quotiented by the fat diagonal, which, by Lemma 2.1, is equivalent to $C_{M^+}(n)$. Alternatively, follow Nick Kuhn's suggestion of proof by intimidation. □

4. Connections with work of Markl

In [7], Markl shows that the configuration spaces for an m-dimensional parallelizable manifold M form a right module over the operad F_m. This is the operad, described by Getzler-Jones [4], formed by the Fulton-MacPherson compactifications of the configuration spaces of points in $\mathbb{R}^m$. Here we suggest a possible relationship between the two right module structures, one over the derivatives of the identity and one over the Fulton-MacPherson operad.

The key to the (potential) relationship is the following construction. Let P be any operad in *unbased* topological spaces (with respect to the symmetric monoidal structure given by the cartesian product). Adding a disjoint basepoint to each of the terms in P we get an operad P_+ in based spaces (with respect to the smash product). There is an obvious map of operads:

$$P_+ \to \underline{S^0}$$

given by collapsing each $P(n)$ to the non-basepoint of S^0. Applying the bar construction and taking Spanier-Whitehead duals we get a map of operads

$$\partial_* I = \mathrm{Map}_*(B(\underline{S^0}), \mathbb{S}) \to \mathrm{Map}_*(B(P_+), \mathbb{S}).$$

We now take P to be equal to F_m. Salvatore showed in [8] that the operad F_m is 'self-dual' in the following sense:

Lemma 4.1 (Salvatore)**.** *Let F_m denote the Fulton-MacPherson operad of compactified configuration spaces for $\mathbb{R}^m$. Then*

$$\mathrm{Map}_*(B(\mathsf{F}_{m+}), \mathbb{S})(n) \simeq \Sigma^{-m(n-1)}\Sigma^\infty \mathsf{F}_m(n)_+.$$

Moreover, these equivalences respect the operad structures on the two sides.

Using this calculation, we obtain a map of operads

$$\partial_* I \to \Sigma^{-m}\mathsf{F}_{m+}$$

where there is a suspension spectrum understood on the right-hand side and $\Sigma^{-m}P$ denotes the operad with

$$(\Sigma^{-m}P)(n) = \Sigma^{-m(n-1)}P(n).$$

We have no particular evidence for the following conjecture.

Conjecture 4.2. *The map $\partial_* I \to \Sigma^{-m}\mathsf{F}_{m+}$ constructed above relates the right module structures formed by the configuration spaces of points in an m-dimensional parallelizable manifold over these two operads, the one given by Proposition 3.1 and the other constructed by Markl.*

References

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