INFINITY-OPERADS AND DAY CONVOLUTION IN GOODWILLIE CALCULUS

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Abstract. We prove two theorems about Goodwillie calculus and use those theorems to describe new models for Goodwillie derivatives of functors between pointed compactly-generated ∞-categories. The first theorem say that the construction of higher derivatives for spectrum-valued functors is a Day convolution of copies of the first derivative construction. The second theorem says that the derivatives of any functor can be realized as natural transformation objects for derivatives of spectrum-valued functors. Together these results allow us to construct an ∞-operad that models the derivatives of the identity functor on any pointed compactly-generated ∞-category.

Our main example is the ∞-category of algebras over a stable ∞-operad, in which case we show that the derivatives of the identity essentially recover the same ∞-operad, making precise a well-known slogan in Goodwillie calculus.

We also describe a bimodule structure on the derivatives of an arbitrary functor, over the ∞-operads given by the derivatives of the identity on the source and target, and we conjecture a chain rule that generalizes previous work of Arone and the author in the case of functors of pointed spaces and spectra.

The fundamental construction of Goodwillie calculus is, for a functor $F : \mathcal{C} \to \mathcal{D}$, a tower of approximations to F that mimics the Taylor series in ordinary calculus. One of the basic principles of this theory is that the fibres of the maps in this tower can be described relatively simply in terms of stable homotopy theory. Indeed, Goodwillie showed that when $\mathcal{C}$ and $\mathcal{D}$ are either the categories of pointed spaces or spectra, the n^{th} homogeneous piece of a functor $F : \mathcal{C} \to \mathcal{D}$ is determined by a single spectrum $\partial_n F$ together with an action of the n^{th} symmetric group Σ_n.

A central question in calculus then is how to reconstruct the Taylor tower of the functor F (and hence, in cases where the tower converges, the functor F itself) from these homogeneous pieces, i.e. from the symmetric sequence $\partial_* F = (\partial_n F)_{n \geq 1}$. In the cases where $\mathcal{C}$ and $\mathcal{D}$ are each either the ∞-category of pointed spaces or of spectra, this was answered in a pair of papers by Greg Arone and the author [1, 2]. We first showed that, for $F : \mathcal{C} \to \mathcal{D}$, the symmetric sequence $\partial_* F$ has the structure of a bimodule over the two operads $\partial_* I_{\mathcal{C}}$ and $\partial_* I_{\mathcal{D}}$ formed by the derivatives of the identity functor on the categories $\mathcal{C}$ and $\mathcal{D}$. We then showed that the resulting adjunction, between the categories of (n-excisive) functors $F : \mathcal{C} \to \mathcal{D}$ and (n-truncated) bimodules, is comonadic, so that an n-excisive functor can be recovered from the action of a certain comonad on the bimodule $\partial_* F$.

In this paper, we extend the first part of that previous work to a broad class of ∞-categories. In particular, we show that the derivatives of the identity functor on any pointed compactly-generated ∞-category form a stable ∞-operad in a natural way, and that the derivatives of any functor form a bimodule over the appropriate ∞-operads. We will review basic facts about ∞-operads in Section 4, and none of the technical details of the theory of ∞-categories is needed before then.

Note that the approach taken in this paper is significantly different from that of [1], and even in the cases of pointed spaces and spectra it gives a new perspective on how the operad structures arise. In particular, this paper provides a new construction of the spectral Lie operad, as an ∞-operad, distinct from the cobar construction described in [6].

One of the differences we encounter in the general case is that the n^{th} layer of the Taylor tower is no longer determined by a single spectrum. This is a consequence of the fact that an arbitrary stable ∞-category, unlike the category of spectra, does not have a single compact generator. Our definition of the derivatives of a functor (given in Section 1) is therefore necessarily more involved.

For us the n^{th} derivative of a functor $F : \mathcal{C} \to \mathcal{D}$ is a diagram of spectra of the form

$$\partial_n F : \mathcal{S}p(\mathcal{C})^n \times \mathcal{S}p(\mathcal{D})^{op} \to \mathcal{S}p$$

that is symmetric in the n copies of $\mathcal{S}p(\mathcal{C})$, and is linear in each variable. Here $\mathcal{S}p(\mathcal{C})$ denotes the stabilization of the ∞-category $\mathcal{C}$ as described by Lurie in [18, 1.4].

When, in addition, $\mathcal{C}$ and $\mathcal{D}$ are each the ∞-category of pointed spaces or spectra, the stabilizations are just $\mathcal{S}p$, the ∞-category of spectra. If F preserves filtered colimits, the resulting symmetric multilinear functor $\mathcal{S}p^n \times \mathcal{S}p^{op} \to \mathcal{S}p$ is determined by its value on the sphere spectrum in each variable. This value recovers the spectrum with Σ_n-action that is usually referred to as the n^{th} derivative of the functor F in this case. To simplify this introduction we suppress the dependence of the derivative on other variables in what follows. More explicit statements in the case of general $\mathcal{C}$ and $\mathcal{D}$ can be found in the main body of the paper.

Our philosophy is to start by focusing on functors $F : \mathcal{C} \to \mathcal{S}p$. Let $\mathcal{F}_{\mathcal{C}}$ denote the ∞-category of those functors of this type that are reduced (i.e. $F(*) \simeq *$) and finitary (preserve filtered colimits). The construction of the n^{th} derivative can then itself be viewed as a functor

$$\partial_n : \mathcal{F}_{\mathcal{C}} \to \mathcal{S}p.$$

Now the ∞-category $\mathcal{F}_{\mathcal{C}}$ has a (non-unital) symmetric monoidal product given by the objectwise smash product of functors, and therefore the category $\mathrm{Fun}(\mathcal{F}_{\mathcal{C}}, \mathcal{S}p)$ of functors $\mathcal{F}_{\mathcal{C}} \to \mathcal{S}p$ has a symmetric monoidal product $\otimes$ given by the *Day convolution* of the objectwise smash product on $\mathcal{F}_{\mathcal{C}}$ and the ordinary smash product on $\wedge$.

Our first main theorem (proved in Section 2) gives a relationship between the functors ∂_n, for different n, in terms of this Day convolution structure.

Theorem 0.1. *Let $\mathcal{C}$ be a pointed compactly-generated ∞-category. Then there is a Σ_n-equivariant equivalence, in the ∞-category* $\mathrm{Fun}(\mathcal{F}_{\mathcal{C}}, \mathcal{S}p)$*, of the form*

$$\partial_n \simeq \partial_1^{\otimes n}.$$

Next we turn to (reduced, finitary) functors $F : \mathcal{C} \to \mathcal{D}$ between two arbitrary pointed compactly-generated ∞-categories. Our second main theorem (proved in Section 3) allows us to identify the derivatives of such a functor F in terms of the derivatives of spectrum-valued functors on $\mathcal{C}$ and $\mathcal{D}$.

Theorem 0.2. *Let $F : \mathcal{C} \to \mathcal{D}$ be a reduced functor that preserves filtered colimits. Then there is a natural equivalence*

$$\partial_n F \simeq \mathrm{Nat}(\partial_1(-), \partial_n(- \circ F))$$

where the right-hand side is the spectrum of natural transformations between two functors of type $\mathcal{F}_{\mathcal{D}} \to \mathcal{S}p$.

Combining Theorems 0.1 and 0.2, we get new models for $\partial_n F$ that can be defined entirely in terms of the first derivative construction for spectrum-valued functors, and Day convolution:

$$\partial_n F \simeq \mathrm{Nat}(\partial_1(-), \partial_1^{\otimes n}(- \circ F)). \tag{0.3}$$

One unanswered question of [1] was whether such models can admit (unital and associative) composition maps of the form

$$\partial_*(G) \circ \partial_*(F) \to \partial_*(GF) \tag{0.4}$$

when F and G are composable, which, in particular, provide the derivatives of the identity functor (or, indeed, any monad) with an operad structure. In the cases of pointed spaces and spectra such a construction has recently been made by Yeakel [21].

The models given in (0.3) permit the construction of composition maps of the form (0.4) and allow us to extend Yeakel's result to a much wider range of ∞-categories (though with this generalization comes constructions that are less explicit at the point-set level). In particular, when F is the identity functor $I_{\mathcal{C}}$ on a pointed compactly-generated ∞-category $\mathcal{C}$, we get

$$\partial_n I_{\mathcal{C}} \simeq \mathrm{Nat}(\partial_1, \partial_1^{\otimes n}).$$

The symmetric sequence $\partial_* I_{\mathcal{C}}$ has an operad structure given by composition of natural transformations, a so-called 'coendomorphism operad' for the object ∂_1 with respect to Day convolution. In a similar way, for $F : \mathcal{C} \to \mathcal{D}$, the derivatives of F form a bimodule over the operads $\partial_* I_{\mathcal{C}}$ and $\partial_* I_{\mathcal{D}}$.

To be more precise, what we get are ∞-operads (in the sense of Lurie [18, 2.1]) and bimodules over those ∞-operads. We give explicit constructions of these objects in Sections 4 and 6. Those constructions rely heavily on some technical constructions with symmetric monoidal ∞-categories: work of Glasman [11] (on Day convolution) and of Barwick-Glasman-Nardin [4] (on opposite symmetric monoidal structures). Combining these two pieces of work, we get a (non-unital) symmetric monoidal ∞-category

$$\mathrm{Fun}(\mathcal{F}_{\mathcal{C}}, \mathcal{S}p)^{op,\otimes}$$

whose underlying ∞-category is the opposite of the ∞-category of functors $\mathcal{F}_{\mathcal{C}} \to \mathcal{S}p$. (In fact, we have to take care over size issues at this point, and replace $\mathcal{F}_{\mathcal{C}}$ with a small symmetric monoidal subcategory, but we will ignore that issue for the rest of this introduction.)

Taking the suboperad of $\mathrm{Fun}(\mathcal{F}_{\mathcal{C}}, \mathcal{S}p)^{op,\otimes}$ generated by ∂_1 then produces an ∞-operad $\mathbb{I}_{\mathcal{C}}^{\otimes}$ that encodes the coendomorphism operad structure on $\partial_* I_{\mathcal{C}}$ described above.

In Section 5, we focus on one principal example of our general theory: the case where $\mathcal{C}$ is the ∞-category of (non-unital) stable algebras over a stable ∞-operad $\mathcal{O}^{\otimes}$. This example includes ∞-categories of structured ring spectra, such as E_n-ring spectra and spectral Lie algebras. Our general result is a calculation of the ∞-operad $\mathbb{I}_{\mathcal{C}}^{\otimes}$ in this case:

Theorem 0.5. *Let $\mathcal{O}^{\otimes}$ be a stable ∞-operad and let $\mathsf{Alg}_{\mathcal{O}^{\otimes}}$ be the ∞-category of non-unital $\mathcal{O}^{\otimes}$-algebras in $\mathcal{S}p$. Then $\mathbb{I}_{\mathsf{Alg}_{\mathcal{O}^{\otimes}}}^{\otimes}$ is (Morita-)equivalent to $\mathcal{O}^{\otimes}$ itself.*

By a Morita equivalence of stable ∞-operads, we mean an equivalence between the ∞-categories of stable algebras over those ∞-operads. We actually show a stronger result: that $\mathbb{I}_{\mathsf{Alg}_{\mathcal{O}^{\otimes}}}^{\otimes}$ contains $\mathcal{O}^{\otimes}$ as a full suboperad, and it is the inclusion of that suboperad that induces an equivalence between ∞-categories of algebras.

Theorem 0.5 verifies a longstanding principle in Goodwillie calculus: that the derivatives of the identity functor on a category of operadic algebras recovers the original operad. For example, this principle can be seen on an arity-wise basis in the work of Harper and Hess [16, 1.14]. Here we promote that principle to a full equivalence of ($\infty-$)operads.

In Section 6 we turn back to the derivatives of an arbitrary (reduced, finitary) functor $F : \mathcal{C} \to \mathcal{D}$ between pointed compactly-generated ∞-categories, and provide a precise construction of the derivatives of F as a bimodule over the ∞-operads $\partial_* I_{\mathcal{C}}$ and $\partial_* I_{\mathcal{D}}$. By a *bimodule* over two ∞-operads, we mean a Δ^1-family of ∞-operads (in the sense of Lurie [18, 2.3.2.10]) that restricts to the given ∞-operads over the endpoints of Δ^1.

For a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, our construction gives rise to a 'composition' map

$$\mathbb{D}_G^{\otimes} \circ_{\mathbb{I}_{\mathcal{D}}^{\otimes}} \mathbb{D}_F^{\otimes} \to \mathbb{D}_{GF}^{\otimes}$$

We conjecture (but do not prove) a Chain Rule, generalizing that of [1], which says that the above map is an equivalence of $(\mathbb{I}_{\mathcal{E}}^{\otimes}, \mathbb{I}_{\mathcal{C}}^{\otimes})$-bimodules.

Technical background. We use ∞-categories, also known as quasicategories, as our basic model for $(\infty, 1)$-categories, yet very little technical knowledge of this theory is required in sections 1-3 of the paper. Our two main results about Goodwillie calculus depend only on basic homotopy theory such as, for example, properties of homotopy limits and colimits. These results could be stated, and proved, in more-or-less exactly the same way in the context of simplicial model categories instead.

In later sections, the theory of ∞-categories, and in particular that of ∞-operads, as developed by Lurie, plays a much more concerted role. We rely heavily on [17] and [18] as references, though we do recall the basic principles of the theory of ∞-operads in Section 4. We require two particularly technical constructions on symmetric monoidal ∞-categories: the Day convolution structure due to Glasman [11]; and the opposite structure due to Barwick-Glasman-Nardin [4]. In fact, to produce bimodules between ∞-operads, we require fibrewise versions of those constructions which we describe in Appendices B and C.

For the initial development of Goodwillie calculus in the context of ∞-categories, we rely on [18, Ch. 6], though the reader will not need any of the technical details of that work.

Notation. We use letters such as $\mathcal{C}, \mathcal{D}$ to stand for pointed compactly-generated ∞-categories, and the symbol $*$ to denote a null object in such. In particular, we have $\mathcal{T}op_*$, the ∞-category of pointed (small) Kan complexes, and $\mathcal{S}p$, the ∞-category of spectra from [18, 1.4.3]. We also make use of the standard adjunction

$$\Sigma^\infty : \mathcal{T}op_* \rightleftarrows \mathcal{S}p : \Omega^\infty.$$

For a pointed ∞-category $\mathcal{C}$, we write $\operatorname{Hom}_{\mathcal{C}}(-,-)$ for (some model of) the pointed simplicial set of maps between two objects of $\mathcal{C}$. If $\mathcal{C}$ is stable, it admits mapping spectra which we denote $\operatorname{Map}_{\mathcal{C}}(-,-)$, so that $\operatorname{Hom}_{\mathcal{C}}(-,-) \simeq \Omega^\infty \operatorname{Map}_{\mathcal{C}}(-,-)$.

A pointed compactly-generated ∞-category $\mathcal{C}$ admits tensors by pointed simplicial sets, which we will write with a smash product symbol. In particular, we have suspensions $\Sigma^L x \simeq S^L \wedge x$ for $x \in \mathcal{C}$. Note that we then have natural equivalences of pointed simplicial sets

$$\Omega^L \operatorname{Hom}_{\mathcal{C}}(x,-) \simeq \operatorname{Hom}_{\mathcal{C}}(\Sigma^L x,-).$$

We will often consider the ∞-category of functors between two other ∞-categories, which we denote in the form $\operatorname{Fun}(\mathcal{C},\mathcal{D})$. When $\mathcal{D} = \mathcal{S}p$, the ∞-category $\operatorname{Fun}(\mathcal{C},\mathcal{D})$ is stable in which case we will write

$$\operatorname{Nat}_{\mathcal{C}}(-,-) := \operatorname{Map}_{\operatorname{Fun}(\mathcal{C},\mathcal{S}p)}(-,-).$$

This plays the role of a spectrum of natural transformations between two such functors.

We usually omit notation for the nerve of a category: for example, $\mathcal{F}\mathrm{in}_*$ denotes the ∞-category given by the nerve of the category of finite pointed sets and pointed maps, and $\mathcal{S}\mathrm{urj}_*$ is the nerve of the category of finite pointed sets and pointed surjections.

When we say limit or colimit, we almost always mean *homotopy* limit or colimit (and we do denote these as holim or hocolim). The exception is when constructing an ∞-category, for example as a pullback of other ∞-categories, in which case we always intend a strict pullback in the category of simplicial sets.

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1. Goodwillie derivatives in infinity-categories

Let $F : \mathcal{C} \to \mathcal{D}$ be a reduced functor between pointed compactly-generated ∞-categories. Such F has a *Taylor tower* constructed in this generality by Lurie [18, 6.1] following Goodwillie's original approach [14]. This is a sequence of functors of the form

$$F \to \cdots \to P_nF \to P_{n-1}F \to \cdots \to P_1F \to P_0F \simeq *$$

where $F \to P_nF$ is initial (up to homotopy) among natural transformations from F to an n-excisive functor. The n^{th} *layer* in the Taylor tower is the fibre

$$D_nF := \mathrm{hofib}(P_nF \to P_{n-1}F)$$

and $D_nF : \mathcal{C} \to \mathcal{D}$ is an n-homogenous functor.

One of Goodwillie's main results provides a classification of homogeneous functors, which shows that the n^{th} layer D_nF can be recovered from a symmetric multilinear functor $\Delta_nF : \mathcal{C}^n \to \mathcal{D}$ (the cross-effect of D_nF, see [18, 6.1.4.14]) by the formula

$$D_nF(X) \simeq \Delta_nF(X, \ldots, X)_{h\Sigma_n}.$$

The symmetric multilinear functor Δ_nF factors, up to equivalence, as

$$\mathcal{C}^n \xrightarrow{\Sigma^{\infty\,n}_{\mathcal{C}}} \mathcal{S}p(\mathcal{C})^n \xrightarrow{\boldsymbol{\Delta}_nF} \mathcal{S}p(\mathcal{D}) \xrightarrow{\Omega^\infty_{\mathcal{D}}} \mathcal{D}$$

where $\boldsymbol{\Delta}_nF$ is another symmetric multilinear functor, and

$$\Sigma^\infty_{\mathcal{C}} : \mathcal{C} \rightleftarrows \mathcal{S}p(\mathcal{C}) : \Omega^\infty_{\mathcal{C}}$$

is the stabilization adjunction for $\mathcal{C}$, see [18, 6.2.3.22].

Definition 1.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a reduced functor between pointed compactly-generated ∞-categories, and let $\mathbf{\Delta}_n F : \mathcal{S}p(\mathcal{C})^n \to \mathcal{S}p(\mathcal{D})$ be as described above. The n^{th} *derivative of* F is the functor

$$\partial_n F : \mathcal{S}p(\mathcal{C})^n \times \mathcal{S}p(\mathcal{D})^{op} \to \mathcal{S}p$$

defined by

$$\partial_n F(X_1, \dots, X_n; Y) := \mathrm{Map}_{\mathcal{S}p(\mathcal{D})}(Y, \mathbf{\Delta}_n F(X_1, \dots, X_n))$$

where $\mathrm{Map}_{\mathcal{S}p(\mathcal{D})}(-,-)$ denotes a mapping spectrum construction for the stable ∞-category $\mathcal{S}p(\mathcal{D})$. In other words, we can think of $\partial_n F$ as the composite of $\mathbf{\Delta}_n F$ with the stable Yoneda embedding for the stable ∞-category $\mathcal{S}p(\mathcal{D})$.

Note that $\partial_n F$ is symmetric multilinear in the $\mathcal{S}p(\mathcal{C})$ variables, and preserves all limits in $\mathcal{S}p(\mathcal{D})^{op}$ (that is, takes colimits in $\mathcal{S}p(\mathcal{D})$ to limits in $\mathcal{S}p$).

Example 1.2. When $\mathcal{C}$ and $\mathcal{D}$ are both either $\mathcal{T}op_*$ or $\mathcal{S}p$, and F preserves filtered colimits, the functor $\partial_n F$ of Definition 1.1 is determined by the single spectrum (with Σ_n-action)

$$\partial_n F(S^0, \dots, S^0; S^0)$$

where S^0 is the sphere spectrum. We write $\partial_n F$ also for this individual spectrum; this is the object typically referred to as *the* n^{th} derivative of F in this case.

Example 1.3. When $\mathcal{D}$ is $\mathcal{T}op_*$ or $\mathcal{S}p$, there is an equivalence

$$\partial_n F(X_1, \dots, X_n; S^0) \simeq \mathbf{\Delta}_n F(X_1, \dots, X_n).$$

We will also write this as $\partial_n F(X_1, \dots, X_n)$. More generally, whenever either $\mathcal{C}$ or $\mathcal{D}$ is $\mathcal{T}op_*$ or $\mathcal{S}p$, we may omit the corresponding arguments of the functor $\partial_n F$, in which case those arguments are assumed to be the sphere spectrum S^0.

2. Derivatives of spectrum-valued functors

We now turn to our first main result, which concerns the derivatives of spectrum-valued functors.

Definition 2.1. Fix a pointed compactly-generated ∞-category $\mathcal{C}$ and let $\mathcal{F}_{\mathcal{C}}$ be the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S}p)$ whose objects are the reduced functors $\mathcal{C} \to \mathcal{S}p$ that preserve filtered colimits. For objects $X_1, \dots, X_n \in \mathcal{S}p(\mathcal{C})$, Example 1.3 says that we have a functor

$$\partial_n(-)(X_1, \dots, X_n) : \mathcal{F}_{\mathcal{C}} \to \mathcal{S}p.$$

The goal of this section is to understand how these functors are related to one another for varying n.

The relationship we are looking for is via a version of Day convolution (see [9]) for such functors $\mathcal{F}_{\mathcal{C}} \to \mathcal{S}p$ with respect to the following symmetric monoidal structures: on $\mathcal{F}_{\mathcal{C}}$ the objectwise smash product of functors; and on $\mathcal{S}p$ the ordinary smash product. Later in the paper, we will work with a symmetric monoidal structure that represents this Day convolution, but for now it is sufficient to describe convolution by its universal property.

Definition 2.2. The *Day convolution* of $\mathsf{A}, \mathsf{B} : \mathcal{F}_{\mathcal{C}} \to \mathcal{S}p$, if it exists, consists of a functor

$$\mathsf{A} \otimes \mathsf{B} : \mathcal{F}_{\mathcal{C}} \to \mathcal{S}p$$

and a natural transformation of functors $\mathcal{F}_{\mathcal{C}} \times \mathcal{F}_{\mathcal{C}} \to \mathcal{S}p$ of the form

$$\alpha : \mathsf{A}(-) \wedge \mathsf{B}(-) \to (\mathsf{A} \otimes \mathsf{B})(- \wedge -)$$

that induces equivalences of mapping spaces

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{F}_{\mathcal{C}}, \mathcal{S}p)}(\mathsf{A} \otimes \mathsf{B}, \mathsf{C}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{F}_{\mathcal{C}} \times \mathcal{F}_{\mathcal{C}}, \mathcal{S}p)}(\mathsf{A}(-) \wedge \mathsf{B}(-), \mathsf{C}(- \wedge -))$$

for an arbitrary functor $\mathsf{C} : \mathcal{F}_{\mathcal{C}} \to \mathcal{S}p$. Note that we use the symbol $\wedge$ on the right-hand side to denote both the smash product of spectra and the objectwise smash product on $\mathcal{F}_{\mathcal{C}}$, as appropriate. We define convolution of more than two functors in a similar way.

Remark 2.3. Definition 2.2 says that Day convolution is a left Kan extension, and it follows that the convolution is unique up to equivalence. In the cases we care about, we will prove existence directly, primarily via Lemma 2.16 below. In Section 4, we will use work of Glasman [11] to construct a (non-unital) symmetric monoidal ∞-category whose monoidal structure represents the Day convolution, at least for functors on a small symmetric monoidal subcategory of $\mathcal{F}_{\mathcal{C}}$.

The main result of this section is the following relationship between the n^{th} and 1^{st} derivative constructions for functors from $\mathcal{C}$ to $\mathcal{S}p$.

Theorem 2.4. *Let $\mathcal{C}$ be a pointed compactly-generated ∞-category, and consider objects $X_1, \dots, X_n \in \mathcal{S}p(\mathcal{C})$. Then there is a natural equivalence*

$$\partial_n(-)(X_1, \dots, X_n) \simeq \partial_1(-)(X_1) \otimes \cdots \otimes \partial_1(-)(X_n)$$

where ∂_n denotes the n^{th} derivative construction for functors $\mathcal{C} \to \mathcal{S}p$, and $\otimes$ denotes the Day convolution of Definition 2.2.

Corollary 2.5. *When $\mathcal{C} = \mathcal{T}op_*$ or $\mathcal{S}p$, taking $X_1 = \cdots = X_n = S^0$ in Theorem 2.4 gives the formula*

$$\partial_n \simeq \partial_1^{\otimes n}.$$

Remark 2.6. The Day convolution cannot be calculated objectwise. In particular, this theorem does *not* imply that the n^{th} derivative of a *particular* functor $F : \mathcal{C} \to \mathcal{S}p$ can be calculated from the first derivative of F (which would clearly be false). Rather it says that $\partial_n F$ can be calculated as a homotopy colimit of the form

$$\partial_n F \simeq \underset{G_1 \wedge \ldots \wedge G_n \to F}{\mathrm{hocolim}} \partial_1 G_1 \wedge \ldots \wedge \partial_1 G_n$$

calculated over the ∞-category of n-tuples of functors $G_1, \dots, G_n$ given a map $G_1 \wedge \ldots \wedge G_n \to F$.

Remark 2.7. Since Day convolution is, in fact, a symmetric monoidal structure, Theorem 2.4 allows us to see that the collection of functors $(\partial_n)_{n\geq 1}$ possesses additional structure. Suppose we define a coloured operad $\mathbb{I}_{\mathcal{C}}$, enriched in $\mathcal{S}p$, with colours given by the objects of $\mathcal{S}p(\mathcal{C})$ and terms

$$\mathbb{I}_{\mathcal{C}}(X_1,\dots,X_n;Y) = \operatorname{Nat}_{\mathcal{F}_{\mathcal{C}}}(\partial_1(-)(Y), \partial_1(-)(X_1)\otimes\cdots\otimes\partial_1(-)(X_n)). \tag{2.8}$$

where $\operatorname{Nat}_{\mathcal{F}_{\mathcal{C}}}(-,-)$ denotes a mapping *spectrum* construction for the stable ∞-category $\operatorname{Fun}(\mathcal{F}_{\mathcal{C}},\mathcal{S}p)$. The operad structure is given by composition of natural transformations. It is then an easy consequence of Theorem 2.4 that the derivatives of any functor $\mathcal{C}\to\mathcal{S}p$ form a right module over the operad $\mathbb{I}_{\mathcal{C}}$. As stated, these operad and module structures are only associative up to homotopy; a more precise definition of $\mathbb{I}_{\mathcal{C}}$ as an ∞-operad will be given in Section 4.

The remainder of this section consists of the proof of Theorem 2.4. This proof relies largely on Goodwillie's identification of the derivative as a multilinearized cross-effect. That is, we have, for $x_1,\dots,x_n\in\mathcal{C}$ and $F:\mathcal{C}\to\mathcal{S}p$:

$$\Delta_n F(x_1,\dots,x_n) \simeq \operatorname*{hocolim}_{L\to\infty} \Omega^{nL}\operatorname{cr}_n F(\Sigma^L x_1,\dots,\Sigma^L x_n). \tag{2.9}$$

where $\Delta_n F$ is the symmetric multilinear functor that classifies the homogeneous functor D_nF. (An ∞-categorical version of this result follows from [18, 6.1.3.23 and 6.1.1.28].)

We also use the fact, extending [3, 3.13], that cross-effects of spectrum-valued functors can be represented as natural transformation objects. To see this, we first need a version of the Yoneda Lemma in this context.

Lemma 2.10. *Let $\mathcal{C}$ be a pointed ∞-category, x an object of $\mathcal{C}$, and $F:\mathcal{C}\to\mathcal{S}p$ a reduced functor. Then there is a natural equivalence of spectra*

$$\operatorname{Nat}_{\mathcal{C}}(\Sigma^\infty \operatorname{Hom}_{\mathcal{C}}(x,-),F(-)) \simeq F(x)$$

where recall that the left-hand side denotes a mapping spectrum for the stable ∞-category $\operatorname{Fun}(\mathcal{C},\mathcal{S}p)$.

Proof. Any functor $F:\mathcal{C}\to\mathcal{S}p$ admits a natural map

$$\operatorname{Hom}_{\mathcal{C}}(x,-)\to\operatorname{Hom}_{\mathcal{S}p}(F(x),F(-))\simeq\Omega^\infty\operatorname{Map}_{\mathcal{S}p}(F(x),F(-))$$

which is basepoint-preserving when F is reduced. This map therefore corresponds via adjunctions to the desired map

$$F(x)\to\operatorname{Nat}_{\mathcal{C}}(\Sigma^\infty\operatorname{Hom}_{\mathcal{C}}(x,-),F(-)).$$

To prove this map is an equivalence of spectra, it is sufficient to show that each induced map

$$\Omega^\infty\Sigma^{-k}F(x)\to\Omega^\infty\Sigma^{-k}\operatorname{Nat}_{\mathcal{C}}(\Sigma^\infty\operatorname{Hom}_{\mathcal{C}}(x,-),F(-))$$

is an equivalence of simplicial sets. We can identify the right-hand side with

$$\operatorname{Hom}_{\operatorname{Fun}(\mathcal{C},\mathcal{S}p)}(\operatorname{Hom}_{\mathcal{C}}(x,-),\Omega^\infty\Sigma^{-k}F(-))$$

and the claim follows from the ordinary Yoneda Lemma. □

We then have the following description of the cross-effects of a reduced functor $F : \mathcal{C} \to \mathcal{S}p$.

Lemma 2.11. *Let $\mathcal{C}$ be a pointed ∞-category. For reduced $F : \mathcal{C} \to \mathcal{S}p$ and objects $x_1, \dots, x_n \in \mathcal{C}$, we have a natural equivalence*

$$\operatorname{cr}_n F(x_1, \dots, x_n) \simeq \operatorname{Nat}_{\mathcal{C}}(\Sigma^\infty \operatorname{Hom}_{\mathcal{C}}(x_1, -) \wedge \dots \wedge \Sigma^\infty \operatorname{Hom}_{\mathcal{C}}(x_n, -), F(-)).$$

Here $\operatorname{Nat}_{\mathcal{C}}(-,-)$ denotes a mapping spectrum for the stable ∞-category $\operatorname{Fun}(\mathcal{C}, \mathcal{S}p)$.

Proof. The case $n = 1$ is Lemma 2.10 since $\operatorname{cr}_1 F \simeq F$ when F is reduced. We describe the case $n = 2$. The general case is virtually identical.

Recall that the n^{th} cross-effect is defined as the total fibre of an n-cube (see [13]): for $n = 2$, this cube takes the form

$$\operatorname{cr}_2 F(x_1, x_2) \simeq \operatorname{thofib} \left(\begin{array}{ccc} F(x_1 \vee x_2) & \longrightarrow & F(x_1) \\ \downarrow & & \downarrow \\ F(x_2) & \longrightarrow & F(*) \end{array} \right).$$

Using Lemma 2.10 we can write the square on the right-hand side here as

$$\begin{array}{ccc} \operatorname{Nat}_{\mathcal{C}}(\Sigma^\infty \operatorname{Hom}_{\mathcal{C}}(x_1 \vee x_2, -), F(-)) & \longrightarrow & \operatorname{Nat}_{\mathcal{C}}(\Sigma^\infty \operatorname{Hom}_{\mathcal{C}}(x_1, -), F(-)) \\ \downarrow & & \downarrow \\ \operatorname{Nat}_{\mathcal{C}}(\Sigma^\infty \operatorname{Hom}_{\mathcal{C}}(x_2, -), F(-)) & \longrightarrow & \operatorname{Nat}_{\mathcal{C}}(\Sigma^\infty \operatorname{Hom}_{\mathcal{C}}(*, -), F(-)). \end{array}$$

Since $\operatorname{Nat}_{\mathcal{C}}(\Sigma^\infty -, F)$ takes colimits (of $\mathcal{T}op_*$-valued functors on $\mathcal{C}$) to limits (of spectra), the total fibre of the above square is equivalent to

$$(*) \qquad \operatorname{Nat}_{\mathcal{C}}(\Sigma^\infty A(-), F(-))$$

where $A(-)$ is the total *cofibre* of the 2-cube of spaces of the form

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{C}}(*, -) & \longrightarrow & \operatorname{Hom}_{\mathcal{C}}(x_1, -) \\ \downarrow & & \downarrow \\ \operatorname{Hom}_{\mathcal{C}}(x_2, -) & \longrightarrow & \operatorname{Hom}_{\mathcal{C}}(x_1 \vee x_2, -) \end{array}$$

which can be written in the form

$$\begin{array}{ccc} * & \longrightarrow & \operatorname{Hom}_{\mathcal{C}}(x_1, -) \\ \downarrow & & \downarrow \\ \operatorname{Hom}_{\mathcal{C}}(x_2, -) & \longrightarrow & \operatorname{Hom}_{\mathcal{C}}(x_1, -) \times \operatorname{Hom}_{\mathcal{C}}(x_2, -). \end{array}$$

But then the total cofibre $A(-)$ is equivalent to the smash product

$$\mathrm{Hom}_{\mathcal{C}}(x_1, -) \wedge \mathrm{Hom}_{\mathcal{C}}(x_2, -)$$

which, together with (*), provides the desired equivalence. For the case of general n, the key observation then is that the total cofibre of an n-cube of pointed spaces of the form

$$\left\{ \prod_{i \in S} A_i \right\}_{S \subseteq \{1,\dots,n\}}$$

is equivalent to the smash product $A_1 \wedge \dots \wedge A_n$. □

It follows from Lemma 2.11 that the terms appearing in the homotopy colimit of (2.9) can also be expressed in terms of natural transformation objects:

$$\begin{aligned} \Omega^{nL} \operatorname{cr}_n(F)(\Sigma^L x_1, \dots, \Sigma^L x_n) &\simeq \Omega^{nL} \mathrm{Nat}_{\mathcal{C}}\left(\bigwedge_{i=1}^n \Sigma^\infty \mathrm{Hom}_{\mathcal{C}}(\Sigma^L x_i, -), F \right) \\ &\simeq \mathrm{Nat}_{\mathcal{C}}\left(\bigwedge_{i=1}^n \Sigma^\infty \Sigma^L \Omega^L \mathrm{Hom}_{\mathcal{C}}(x_i, -), F \right) \end{aligned} \tag{2.12}$$

where the first equivalence is that of Lemma 2.11, and the second is built from several instances of the adjunction (Σ^L, Ω^L).

It remains to identify how these equivalences interact with the maps in the colimit in (2.9).

Lemma 2.13. *For reduced $F : \mathcal{C} \to \mathcal{S}p$ and objects $x_1, \dots, x_n$, the following diagram of spectra commutes up to equivalence:*

$$\begin{array}{ccc} \Omega^{nL} \operatorname{cr}_n F(\Sigma^L x_1, \dots, \Sigma^L x_n) & \xrightarrow{t_{1,\dots,1}(\operatorname{cr}_n F)} & \Omega^{n(L+1)} \operatorname{cr}_n F(\Sigma^{L+1} x_1, \dots, \Sigma^{L+1} x_n) \\ \sim\downarrow & & \downarrow\sim \\ \mathrm{Nat}_{\mathcal{C}}\left(\bigwedge_{i=1}^n \Sigma^\infty \Sigma^L \Omega^L \mathrm{Hom}_{\mathcal{C}}(x_i, -), F \right) & \xrightarrow{\epsilon^*} & \mathrm{Nat}_{\mathcal{C}}\left(\bigwedge_{i=1}^n \Sigma^\infty \Sigma^{L+1} \Omega^{L+1} \mathrm{Hom}_{\mathcal{C}}(x_i, -), F \right) \end{array}$$

where the vertical maps are the equivalences of (2.12), the top horizontal map is (the multivariable version of) the stabilization map t_1 appearing in Goodwillie's construction of the linearization of a functor (see [12, 1.10] *or* [18, 6.1.1.27]*), and the bottom horizontal map is that induced by the counit*

$$\epsilon : \Sigma^{L+1}\Omega^{L+1} = \Sigma^L(\Sigma\Omega)\Omega^L \to \Sigma^L\Omega^L.$$

Proof. We illustrate with the case $n = 1$. The general case is similar. Since $\mathrm{cr}_1 F \simeq F$ for F reduced, our diagram takes the form

$$\begin{array}{ccc}
\Omega^L F \Sigma^L x_1 & \xrightarrow{t_1 F} & \Omega^{L+1} F \Sigma^{L+1} x_1 \\
\downarrow \sim & & \downarrow \sim \\
\Omega^L \operatorname{Nat}_{\mathcal{C}}(\Sigma^\infty \operatorname{Hom}_{\mathcal{C}}(\Sigma^L x_1, -), F) & \xrightarrow{\epsilon^*} & \Omega^L \operatorname{Nat}_{\mathcal{C}}(\Sigma^\infty \Sigma\Omega \operatorname{Hom}_{\mathcal{C}}(\Sigma^L x_1, -), F) \\
\downarrow \sim & & \downarrow \sim \\
\operatorname{Nat}_{\mathcal{C}}(\Sigma^\infty \Sigma^L \Omega^L \operatorname{Hom}_{\mathcal{C}}(x_1, -), F(-)) & \xrightarrow{\epsilon^*} & \operatorname{Nat}_{\mathcal{C}}(\Sigma^\infty \Sigma^{L+1} \Omega^{L+1} \operatorname{Hom}_{\mathcal{C}}(x_1, -), F(-))
\end{array}$$

where the bottom square commutes by naturality of ϵ, and the top square is (Ω^L applied to) the $L = 0$ case of the Lemma, with x_1 replaced by $\Sigma^L x_1$. It thus is sufficient to consider the case $L = 0$.

To do this, first recall how the map $t_1 F$ is constructed. Let $\mathbf{D}$ denote the diagram

$$\begin{array}{ccc}
 & & I \\
 & & \downarrow \\
I & \longrightarrow & S^1
\end{array}$$

where I is a closed interval with one of its endpoints as the basepoint, and the two maps are the inclusions into the two halves of the circle S^1.

Two copies of the inclusion $S^0 \to I$ form a cone over $\mathbb{D}$ and induce the horizontal (and diagonal) maps in the following commutative diagram:

$$\begin{array}{ccc}
F(x_1) & \longrightarrow & \operatorname*{holim}_{A \in \mathbf{D}} F(A \wedge x_1) \\
\sim \downarrow Y_{x_1} & & \sim \downarrow \operatorname{holim} Y_{A \wedge x_1} \\
\operatorname{Nat}_{\mathcal{C}}(\Sigma^\infty \operatorname{Hom}_{\mathcal{C}}(x_1, -), F) & \longrightarrow & \operatorname*{holim}_{A \in \mathbf{D}} \operatorname{Nat}_{\mathcal{C}}(\Sigma^\infty \operatorname{Hom}_{\mathcal{C}}(A \wedge x_1, -), F) \\
 & \searrow & \downarrow \sim \\
 & & \operatorname{Nat}_{\mathcal{C}}(\Sigma^\infty \operatorname*{hocolim}_{A \in \mathbf{D}} \operatorname{Hom}_{\mathcal{C}}(x_1, -)^A, F)
\end{array} \tag{2.14}$$

where Y_x denotes the stable Yoneda embedding from Lemma 2.10 applied with object x, and the right-hand bottom vertical map is a canonical equivalence involving the tensoring adjunction for objects in $\mathcal{C}$.

We now argue that the right-hand column of (2.14) can be identified with the desired map, by observing that since I is contractible, each homotopy limit/colimit can be written as a loop-space/suspension. That is, we have the following commutative diagram in which the horizontal maps are induced by the equivalences $* \xrightarrow{\sim} I$:

$$\begin{array}{ccc}
\Omega F \Sigma x_1 & \xrightarrow{\sim} & \operatorname{holim}_{A \in \mathbf{D}} F(A \wedge x_1) \\
\sim \downarrow \Omega Y_{\Sigma x_1} & & \sim \downarrow \operatorname{holim} Y_{A \wedge x_1} \\
\Omega \operatorname{Nat}_{\mathcal{C}}(\Sigma^\infty \operatorname{Hom}_{\mathcal{C}}(\Sigma x_1, -), F) & \xrightarrow{\sim} & \operatorname{holim}_{A \in \mathbf{D}} \operatorname{Nat}_{\mathcal{C}}(\Sigma^\infty \operatorname{Hom}_{\mathcal{C}}(A \wedge x_1, -), F) \\
\downarrow \sim & & \downarrow \sim \\
\operatorname{Nat}_{\mathcal{C}}(\Sigma^\infty \Sigma\Omega \operatorname{Hom}_{\mathcal{C}}(x_1, -), F) & \xrightarrow{\sim} & \operatorname{Nat}_{\mathcal{C}}(\Sigma^\infty \operatorname{hocolim}_{A \in \mathbf{D}} \operatorname{Hom}_{\mathcal{C}}(x_1, -)^A, F)
\end{array}$$

The top map of (2.14) is precisely $t_1 F$, and it is easy to identify the bottom map with ϵ^* when combined with the bottom map in the diagram above. □

Taking homotopy colimits as $L \to \infty$ over the diagrams in Lemma 2.13, and applying (2.9), we therefore get the following result.

Proposition 2.15. *For reduced $F : \mathcal{C} \to \mathcal{S}p$ and objects $x_1, \dots, x_n \in \mathcal{C}$, there is an equivalence*

$$\Delta_n F(x_1, \dots, x_n) \simeq \operatorname{hocolim}_{L \to \infty} \operatorname{Nat}_{\mathcal{C}} \left(\bigwedge_{i=1}^{n} \Sigma^\infty \Sigma^L \Omega^L \operatorname{Hom}_{\mathcal{C}}(x_i, -), F \right)$$

where the maps in the homotopy colimit are induced by the counit map $\epsilon : \Sigma\Omega \to I$.

We also require the following result about Day convolution of representable functors.

Lemma 2.16. *For $F_1, \dots, F_n \in \mathcal{F}_{\mathcal{C}}$, we have an equivalence:*

$$\operatorname{Nat}_{\mathcal{C}}(F_1 \wedge \dots \wedge F_n, -) \simeq \operatorname{Nat}_{\mathcal{C}}(F_1, -) \otimes \cdots \otimes \operatorname{Nat}_{\mathcal{C}}(F_n, -).$$

Proof. We describe the case $n = 2$. The general case is virtually identical. According to Definition 2.2, we first have to produce a natural transformation

$$\alpha : \operatorname{Nat}_{\mathcal{C}}(F_1, -) \wedge \operatorname{Nat}_{\mathcal{C}}(F_2, -) \to \operatorname{Nat}_{\mathcal{C}}(F_1 \wedge F_2, - \wedge -)$$

which we do by taking the smash product of natural transformations.

We then have to show that α induces equivalences

$$(*) \qquad \begin{array}{c}
\operatorname{Hom}_{\operatorname{Fun}(\mathcal{F}_{\mathcal{C}}, \mathcal{S}p)}(\operatorname{Nat}_{\mathcal{C}}(F_1 \wedge F_2, -), \mathsf{A}) \\
\downarrow \\
\operatorname{Hom}_{\operatorname{Fun}(\mathcal{F}_{\mathcal{C}} \times \mathcal{F}_{\mathcal{C}}, \mathcal{S}p)}(\operatorname{Nat}_{\mathcal{C}}(F_1, -) \wedge \operatorname{Nat}_{\mathcal{C}}(F_2, -), \mathsf{A}(- \wedge -))
\end{array}$$

for arbitrary $\mathsf{A} : \mathcal{F}_{\mathcal{C}} \to \mathcal{S}p$.

First note that since $\operatorname{Nat}_{\mathcal{C}}(F_1 \wedge F_2, -)$ and $\operatorname{Nat}_{\mathcal{C}}(F_1, -) \wedge \operatorname{Nat}_{\mathcal{C}}(F_2, -)$ are reduced, it is sufficient to prove (*) is an equivalence when A is reduced. (This is because any natural transformation out of a reduced functor between pointed ∞-categories factors, up to equivalence, via the universal reduction of its target.)

Now notice that a functor of the form $\mathrm{Nat}_{\mathcal{C}}(G,-)$ is linear and hence is equivalent to the linearization of

$$\Sigma^\infty\Omega^\infty \mathrm{Nat}_{\mathcal{C}}(G,-) \simeq \Sigma^\infty \mathrm{Hom}_{\mathcal{F}_{\mathcal{C}}}(G,-).$$

We therefore have an equivalence

$$\begin{aligned} \mathrm{Nat}_{\mathcal{C}}(G,-) &\simeq \operatorname*{hocolim}_{k\to\infty} \Sigma^{-k}\Sigma^\infty \mathrm{Hom}_{\mathcal{F}_{\mathcal{C}}}(G,\Sigma^k(-)) \\ &\simeq \operatorname*{hocolim}_{k\to\infty} \Sigma^{-k}\Sigma^\infty \mathrm{Hom}_{\mathcal{F}_{\mathcal{C}}}(\Sigma^{-k}G,-). \end{aligned}$$

Similarly, the natural transformation α can be identified with the map

$$\begin{array}{c} \operatorname*{hocolim}_{k\to\infty} \Sigma^{-k}\Sigma^\infty \mathrm{Hom}_{\mathcal{F}_{\mathcal{C}}}(\Sigma^{-k}F_1\wedge F_2, -\wedge-) \\ \uparrow \\ \operatorname*{hocolim}_{k_1,k_2\to\infty} \Sigma^{-k_1-k_2}\Sigma^\infty \mathrm{Hom}_{\mathcal{F}_{\mathcal{C}}}(\Sigma^{-k_1}F_1,-)\wedge \mathrm{Hom}_{\mathcal{F}_{\mathcal{C}}}(\Sigma^{-k_2}F_2,-) \end{array}$$

given by inclusion into the term with $k=k_1+k_2$, and therefore, by the Yoneda Lemma (2.10), the map (*) is equivalent to:

$$\begin{array}{c} \operatorname*{holim}_{k\to\infty} \Omega^\infty\Sigma^k \mathsf{A}(\Sigma^{-k}F_1\wedge F_2) \\ \downarrow \\ \operatorname*{holim}_{k_1,k_2\to\infty} \Omega^\infty\Sigma^{k_1+k_2}\mathsf{A}(\Sigma^{-k_1}F_1\wedge\Sigma^{-k_2}F_2) \end{array}$$

induced by projecting onto the term $k=k_1+k_2$. This map is an equivalence since the diagonal map $\mathbb{N}\to\mathbb{N}\times\mathbb{N}$ is final. □

Theorem 2.4 is now a fairly simple consequence of Proposition 2.15 and Lemma 2.16, though we have to be careful about how we extend to $X_1,\dots,X_n\in\mathcal{S}p(\mathcal{C})$.

Proof of Theorem 2.4. We have to prove that for $X_1,\dots,X_n\in\mathcal{S}p(\mathcal{C})$, the functor $\partial_n(-)(X_1,\dots,X_n)$ is a Day convolution of the form

$$\partial_1(-)(X_1)\otimes\cdots\otimes\partial_1(-)(X_n).$$

First suppose that $X_i=\Sigma^\infty_{\mathcal{C}}x_i$ where $x_1,\dots,x_n$ are compact objects in $\mathcal{C}$. In this case, we can apply Lemma 2.16 with the functors $F_i=\Sigma^\infty\Sigma^L\Omega^L\mathrm{Hom}_{\mathcal{C}}(x_i,-)$ (which are in $\mathcal{F}_{\mathcal{C}}$ since each x_i is compact).

It follows easily from Definition 2.2 that the Day convolution commutes with colimits in each variable, so we can take the homotopy colimit as $L\to\infty$ of the result of Lemma 2.16 and, using Proposition 2.15, we get

$$(*)\qquad \Delta_n(-)(x_1,\dots,x_n)\simeq\Delta_1(-)(x_1)\otimes\cdots\otimes\Delta_1(-)(x_n).$$

Recalling that $\partial_n F(\Sigma^\infty x_1, \dots, \Sigma^\infty x_n) \simeq \Delta_n F(x_1, \dots, x_n)$, we see that (*) is precisely the desired equivalence in this case.

Next, note that arbitrary objects $x_1, \dots, x_n$ in the compactly-generated ∞-category $\mathcal{C}$ can be written as filtered colimits of compact objects. We can therefore recover the case of general $x_1, \dots, x_n$ from (*) since again the Day convolution commutes with filtered colimits.

Finally, consider the case of general $X_1, \dots, X_n \in \mathcal{S}p(\mathcal{C})$. The linearization of the functor $\Sigma^\infty_{\mathcal{C}} \Omega^\infty_{\mathcal{C}}$ is equivalent to the identity functor on $\mathcal{S}p(\mathcal{C})$, that is:

$$X_i \simeq P_1(\Sigma^\infty_{\mathcal{C}} \Omega^\infty_{\mathcal{C}})(X_i) \simeq \operatorname*{hocolim}_{k \to \infty} \Omega^k \Sigma^\infty_{\mathcal{C}} \Omega^\infty_{\mathcal{C}} \Sigma^k X_i.$$

In other words, an arbitrary object of $\mathcal{S}p(\mathcal{C})$ can be built from suspension spectrum objects by filtered colimit and desuspension, both of which commute with the Day convolution. We thus obtain the case of general $X_1, \dots, X_n$ from that of suspension spectrum objects. □

3. Derivatives of arbitrary functors

In this section we describe models for the derivatives of an arbitrary (reduced) functor $F : \mathcal{C} \to \mathcal{D}$ between pointed compactly-generated ∞-categories. In particular, we deduce that the terms in the coloured operad $\mathbb{I}_{\mathcal{C}}$ described in Remark 2.7 are given by the derivatives of the identity functor on $\mathcal{C}$. This claim is a consequence of the following theorem.

Theorem 3.1. *For reduced $F : \mathcal{C} \to \mathcal{D}$, $X_1, \dots, X_n \in \mathcal{S}p(\mathcal{C})$ and $Y \in \mathcal{S}p(\mathcal{D})$, we have*

$$\partial_n F(X_1, \dots, X_n; Y) \simeq \mathrm{Nat}_{\mathcal{F}_{\mathcal{D}}}(\partial_1(-)(Y), \partial_n(- \circ F)(X_1, \dots, X_n)).$$

Before giving the proof of 3.1, let us construct the map that realizes this equivalence. That map is based on a natural transformation

$$c : \mathbf{\Delta}_1 G(\mathbf{\Delta}_n F(X_1, \dots, X_n)) \to \mathbf{\Delta}_n(GF)(X_1, \dots, X_n) \tag{3.2}$$

which we now define for $F : \mathcal{C} \to \mathcal{D}$ reduced and $G : \mathcal{D} \to \mathcal{S}p$ reduced and preserving filtered colimits.

Definition 3.3. With F and G as above, there is a natural transformation of functors $\mathcal{C}^n \to \mathcal{S}p$

$$c' : G \operatorname{cr}_n F \to \operatorname{cr}_n(GF)$$

coming from the definition of the cross-effect as a total homotopy fibre. Applying the $(1, \dots, 1)$-excisive approximation to c', we get a map of symmetric multilinear functors

$$c : P_{1,\dots,1}(G \operatorname{cr}_n F) \to P_{1,\dots,1}(\operatorname{cr}_n(GF))$$

which is precisely of the desired form (3.2).

Definition 3.4. For any reduced $G : \mathcal{D} \to \mathcal{S}p$, the functor $\mathbf{\Delta}_1 G = \partial_1 G : \mathcal{S}p(\mathcal{D}) \to \mathcal{S}p$ is, by definition, linear, and hence enriched over $\mathcal{S}p$, at least up to homotopy. In other words we have natural maps

$$\mathrm{Map}_{\mathcal{S}p(\mathcal{D})}(Y, \mathbf{\Delta}_n F(X_1, \dots, X_n)) \to \mathrm{Nat}_{\mathcal{F}_{\mathcal{D}}}(\mathbf{\Delta}_1(-)(Y), \mathbf{\Delta}_1(-)(\mathbf{\Delta}_n F(X_1, \dots, X_n))).$$

Composing with our map c from (3.2) we get

$$\mathrm{Map}_{\mathcal{S}p(\mathcal{D})}(Y, \mathbf{\Delta}_n F(X_1, \dots, X_n)) \to \mathrm{Nat}_{\mathcal{F}_{\mathcal{D}}}(\mathbf{\Delta}_1(-)(Y), \mathbf{\Delta}_n(- \circ F)(X_1, \dots, X_n)) \tag{3.5}$$

which is precisely the form of the equivalence appearing in Theorem 3.1.

Proof of Theorem 3.1. First note that each side of the desired equivalence commutes with desuspension and filtered colimits in the variable Y. The argument in the proof of 2.4 then implies it is sufficient to consider the case $Y = \Sigma^\infty_{\mathcal{D}} y$ for some compact object y in the compactly-generated ∞-category $\mathcal{D}$.

Using Proposition 2.15, the right-hand side of the desired equivalence can then be written in the form

$$\operatorname*{holim}_{L \to \infty} \mathrm{Nat}_{\mathcal{F}_{\mathcal{D}}}(\mathrm{Nat}_{\mathcal{D}}(\Sigma^\infty \Sigma^L \Omega^L \mathrm{Hom}_{\mathcal{D}}(y, \bullet), -), \partial_n(- \circ F)(X_1, \dots, X_n))$$

which, by a stable version of the Yoneda Lemma [19, 6.4], is equivalent to

$$\operatorname*{holim}_{L \to \infty} \partial_n(\Sigma^\infty \Sigma^L \Omega^L \mathrm{Hom}_{\mathcal{D}}(y, F))(X_1, \dots, X_n).$$

On the other hand, for the left-hand side of the desired result, we have an equivalence

$$\partial_n F(X_1, \dots, X_n; \Sigma^\infty_{\mathcal{D}} y) \simeq \partial_n(\mathrm{Hom}_{\mathcal{D}}(y, F))(X_1, \dots, X_n)$$

which follows from the fact that

$$\mathrm{Hom}_{\mathcal{D}}(y, \Delta_n F) \simeq \Delta_n(\mathrm{Hom}_{\mathcal{D}}(y, F))$$

for a compact object $y \in \mathcal{D}$.

It is now sufficient to show that, for reduced $G : \mathcal{C} \to \mathcal{T}op_*$, there is a natural equivalence

$$\alpha : \partial_n G \xrightarrow{\sim} \operatorname*{holim}_{L \to \infty} \partial_n(\Sigma^\infty \Sigma^L \Omega^L G) \tag{3.6}$$

where the map α has components given by

$$\mathbf{\Delta}_n(G) \simeq \mathbf{\Delta}_1(\Sigma^\infty \Sigma^L \Omega^L)\mathbf{\Delta}_n(G) \xrightarrow{c} \mathbf{\Delta}_n(\Sigma^\infty \Sigma^L \Omega^L G)$$

with c as in (3.2). This claim contains the real substance of the result we are trying to prove, and it occupies the majority of our effort here.

We first show that α is an equivalence when $G = \Omega^\infty \mathbb{G}$ for some $\mathbb{G} : \mathcal{C} \to \mathcal{S}p$ (in which case note that $\partial_n G \simeq \partial_n \mathbb{G}$). Then there is a map

$$\beta : \operatorname*{holim}_{L \to \infty} \partial_n(\Sigma^\infty \Sigma^L \Omega^L \Omega^\infty \mathbb{G}) \to \partial_n \mathbb{G}$$

given by projection onto the $L = 0$ term followed by the counit of the adjunction $(\Sigma^\infty, \Omega^\infty)$. It is easy to check from the definitions that $\beta\alpha$ is equivalent to the identity. It is therefore sufficient to show that β is an equivalence.

For this, we need to use an instance of the chain rule for spectrum-valued functors which tells us that there is an equivalence

$$\partial_n(\Sigma^\infty\Sigma^L\Omega^L\Omega^\infty\mathbb{G}) \simeq \prod_{\mathsf{P}(n)} \partial_k(\Sigma^\infty\Sigma^L\Omega^L\Omega^\infty) \wedge \partial_{n_1}\mathbb{G} \wedge \ldots \wedge \partial_{n_k}\mathbb{G}$$

where $\mathsf{P}(n)$ is the set of unordered partitions of the set $\{1,\ldots,n\}$, where $n_1,\ldots,n_k$ denote the sizes of the pieces of a partition, and we have suppressed the dependence on variables $X_1,\ldots,X_n \in \mathcal{S}p(\mathcal{C})$ for the sake of readability. This result is a generalization of the main theorem of [7] with a similar proof. Details are provided in Appendix A.

The source of the map β thus splits as

$$(*) \qquad \prod_{\mathsf{P}(n)} \operatorname*{holim}_{L\to\infty} [\partial_k(\Sigma^\infty\Sigma^L\Omega^L\Omega^\infty) \wedge \partial_{n_1}\mathbb{G} \wedge \ldots \wedge \partial_{n_k}\mathbb{G}]$$

and β is given by projection onto the term corresponding to the indiscrete partition, i.e. with $k = 1$. (Notice that in this term all the maps in the inverse system are equivalences and the homotopy limit is just $\partial_n\mathbb{G}$.)

A standard calculation shows that $\partial_k(\Sigma^\infty\Sigma^L\Omega^L\Omega^\infty) \simeq S^{-L(k-1)}$. The maps in the inverse systems in (*) are induced by the counit $\Sigma\Omega \to I$ via maps

$$\partial_k(\Sigma^\infty\Sigma^{L+1}\Omega^{L+1}\Omega^\infty) \to \partial_k(\Sigma^\infty\Sigma^L\Omega^L\Omega^\infty)$$

and hence are trivial when $k > 1$ for dimension reasons. It follows that the homotopy limits appearing in (*) are trivial when $k > 1$, and hence that the projection map β is an equivalence. This completes the proof that the map α is an equivalence when $G = \Omega^\infty\mathbb{G}$.

Now consider arbitrary reduced $G : \mathcal{C} \to \mathcal{T}op_*$. There is a commutative diagram

$$\begin{array}{ccc}
\partial_n G & \longrightarrow & \operatorname{Tot} \partial_n(\Omega^\infty(\Sigma^\infty\Omega^\infty)^\bullet\Sigma^\infty G) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \operatorname{Tot}\alpha} \\
\operatorname*{holim}_{L\to\infty} \partial_n(\Sigma^\infty\Sigma^L\Omega^L G) & \longrightarrow & \operatorname{Tot} \operatorname*{holim}_{L\to\infty} \partial_n(\Sigma^\infty\Sigma^L\Omega^L\Omega^\infty(\Sigma^\infty\Omega^\infty)^\bullet\Sigma^\infty G)
\end{array}$$

where Tot denotes the totalization of cosimplicial spectra which are built from the $(\Sigma^\infty, \Omega^\infty)$ adjunction. The horizontal maps are equivalences by induction on the Taylor tower of G (by the argument of [1, 4.1.1] and using the fact that Tot commutes with holim), and the right-hand vertical map is an equivalence by the case already considered. Therefore the map α is an equivalence for arbitrary G. This completes the proof of Theorem 3.1. □

Corollary 3.7. *For any pointed compactly-generated ∞-category $\mathcal{C}$, we have*

$$\partial_n I_{\mathcal{C}}(X_1,\ldots,X_n;Y) \simeq \operatorname{Nat}_{\mathcal{F}_{\mathcal{C}}}(\partial_1(-)(Y), \partial_n(-)(X_1,\ldots,X_n)).$$

In particular, this identifies the terms of the coloured operad $\mathbb{I}_{\mathcal{C}}$ described in Remark 2.7.

Example 3.8. When $\mathcal{C} = \mathcal{D} = \mathcal{T}op_*$, we have

$$\partial_n F \simeq \mathrm{Nat}(\partial_1, \partial_n(- \circ F))$$

and, in particular,

$$\partial_n I_{\mathcal{C}} \simeq \mathrm{Nat}(\partial_1, \partial_n) \simeq \mathrm{Nat}(\partial_1, \partial_1^{\otimes n}).$$

In other words, the derivatives of the identity functor on $\mathcal{T}op_*$ form the coendomorphism operad of the functor $\partial_1 : \mathrm{Fun}(\mathcal{T}op_*, \mathcal{S}p) \to \mathcal{S}p$ with respect to Day convolution. In [6] an operad structure on these derivatives was constructed by taking the Koszul dual of the commutative operad in spectra. It is not obvious that these two operad structures on $\partial_* I_{\mathcal{T}op_*}$ are equivalent, though both depend on the cosimplicial resolution of the identity functor via the adjunction $(\Sigma^\infty, \Omega^\infty)$, which makes a connection between them plausible.

Remark 3.9. The key part of the proof of Theorem 3.1 was the construction of the equivalence

$$\alpha : \partial_n G \xrightarrow{\sim} \operatorname*{holim}_{L \to \infty} \partial_n(\Sigma^\infty \Sigma^L \Omega^L G)$$

for a functor $G : \mathcal{C} \to \mathcal{T}op_*$. In particular, when $G = I_{\mathcal{T}op_*}$, we get

$$\partial_* I_{\mathcal{T}op_*} \simeq \operatorname*{holim}_{L \to \infty} \partial_*(\Sigma^\infty \Sigma^L \Omega^L).$$

The terms in the homotopy limit on the right-hand side turn out to be equivalent to the operads $\mathsf{K}(\mathsf{E}_L)$ given by the Koszul duals of the stable little L-discs operad, themselves equivalent to desuspensions of those little disc operads by [8], and this formula expresses $\partial_* I_{\mathcal{T}op_*}$ as the inverse limit of a 'pro-operad'. Similarly, we have an equivalence

$$\partial_* I_{\mathcal{S}p} \simeq \partial_* \Omega^\infty \simeq \operatorname*{holim}_{L \to \infty} \partial_*(\Sigma^\infty \Sigma^L \Omega^L \Omega^\infty) \simeq \operatorname*{holim}_{L \to \infty} \mathbf{S}^{-L}$$

which expresses $\partial_* I_{\mathcal{S}p}$ as the inverse limit of a pro-operad whose components are certain operads $\mathbf{S}^{-L}$ formed by desuspensions of the sphere operad.

In [3], Arone and the author showed that these two pro-operads classify the Taylor towers of functors $\mathcal{T}op_* \to \mathcal{S}p$ and $\mathcal{S}p \to \mathcal{S}p$ respectively. We believe that an analogous pro-operad can be constructed for any pointed, compactly-generated ∞-category $\mathcal{C}$. The inverse limit of this pro-operad is equivalent to the operad $\partial_* I_{\mathcal{C}}$ and modules over the pro-operad should classify the Taylor towers of functors $\mathcal{C} \to \mathcal{S}p$.

Remark 3.10. Theorem 3.1 provides models of the derivatives of a functor $F : \mathcal{C} \to \mathcal{D}$ that admit natural composition maps in the following sense. Define a collection $\mathbb{D}_F$ of spectra by

$$\mathbb{D}_F(X_1, \dots, X_n; Y) := \mathrm{Nat}_{\mathcal{F}_{\mathcal{D}}}(\partial_1(-)(Y), (\partial_1(-)(X_1) \otimes \dots \otimes \partial_1(-)(X_n))(- \circ F))$$

for $X_1, \dots, X_n \in \mathcal{C}$ and $Y \in \mathcal{D}$. Notice that $\mathbb{D}_{I_{\mathcal{C}}}$ is the same collection of spectra as the coloured operad $\mathbb{I}_{\mathcal{C}}$.

Now suppose we have reduced functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ that preserve filtered colimits. Then we can build maps of the form

$$\begin{array}{c} \mathbb{D}_G(Y_1, \dots, Y_k; Z) \wedge \mathbb{D}_F(\underline{X}_1; Y_1) \wedge \dots \wedge \mathbb{D}_F(\underline{X}_k; Y_k) \\ \big\downarrow \\ \mathbb{D}_{GF}(\underline{X}_1, \dots, \underline{X}_k; Z) \end{array}$$

where $Z \in \mathcal{E}$, $Y_1, \dots, Y_k \in \mathcal{D}$ and each $\underline{X}_i$ is a sequence of objects in $\mathcal{C}$. In particular, the derivatives $\mathbb{D}_F$ form a bimodule over the operads $\mathbb{I}_{\mathcal{C}}$ and $\mathbb{I}_{\mathcal{D}}$ described in Remark 2.7, at least up to homotopy. This structure will be made precise, in the context of ∞-operads, in Section 6.

4. Stable infinity-operads and derivatives of the identity

In this section we provide a formal definition of the operad $\mathbb{I}_{\mathcal{C}}$ of Remark 2.7 in the context of Lurie's theory of ∞-operads. Here is an outline of our main construction.

We start by describing a symmetric monoidal ∞-category that represents the objectwise smash product of functors $\mathcal{C} \to \mathcal{S}p$, and hence the desired monoidal product on $\mathcal{F}_{\mathcal{C}}$. Then we turn to the Day convolution, using the work of Glasman [11] to describe a symmetric monoidal ∞-category that represents the convolution of functors $\mathcal{F}_{\mathcal{C}} \to \mathcal{S}p$.

Some care is needed here because the ∞-category $\mathcal{F}_{\mathcal{C}}$ is not small. However, it is generated under filtered colimits by a small symmetric monoidal subcategory $\mathcal{F}^{\omega}_{\mathcal{C}}$. We construct a symmetric monoidal ∞-category $\mathrm{Fun}(\mathcal{F}^{\omega}_{\mathcal{C}}, \mathcal{S}p)^{\otimes}$ that represents the Day convolution of functors $\mathcal{F}^{\omega}_{\mathcal{C}} \to \mathcal{S}p$, and note that the proof of Theorem 2.4 carries over to this context.

As Remark 2.7 shows, we are interested in morphisms *into* the Day convolution rather than out of it, so we next apply work of Barwick, Glasman and Nardin [4] to construct a symmetric monoidal ∞-category $\mathrm{Fun}(\mathcal{F}^{\omega}_{\mathcal{C}}, \mathcal{S}p)^{op,\otimes}$ that represents the *opposite* symmetric monoidal structure to that of Day convolution.

Finally, in Definition 4.19, we restrict to the full subcategory of $\mathrm{Fun}(\mathcal{F}^{\omega}_{\mathcal{C}}, \mathcal{S}p)^{op,\otimes}$ generated by those objects of the form $\partial_1(-)(X)$ for $X \in \mathcal{S}p(\mathcal{C})$. The resulting ∞-operad $\mathbb{I}^{\otimes}_{\mathcal{C}}$ is then a precise version of the operad described informally in Remark 2.7.

We should also note that all the ∞-operads appearing in our work, including the symmetric monoidal structures, will be *non-unital* in the sense that they do not encode unit objects. We start our description of these constructions by recalling the basic theory of ∞-operads from [18] with some slight adjustment to take into account our focus on the non-unital case. Note that the language of (co)cartesian edges and fibrations from [17, 2.4] will be used heavily in the remainder of this paper.

Definition 4.1. Let $\mathcal{S}\mathrm{urj}_*$ denote the category whose objects are pointed finite sets and whose morphisms are surjections which preserve the basepoint. We write $\langle n\rangle := \{*, 1, \dots, n\}$. A morphism in $\mathcal{S}\mathrm{urj}_*$ is *inert* if the inverse image of every non-basepoint contains exactly one element. For example, let $\rho_i : \langle n\rangle \to \langle 1\rangle$ denote the inert morphism with $\rho_i(i) = 1$ and $\rho_i(j) = *$ for $j \neq i$. A morphism is *active* if the inverse image of the basepoint is only the basepoint.

A *non-unital* ∞*-operad* is a map of ∞-categories of the form

$$p : \mathcal{O}^{\otimes} \to \mathcal{S}\mathrm{urj}_*$$

that satisfies the following conditions:

(1) for every object $\underline{X} \in \mathcal{O}^{\otimes}$, every inert morphism α in $\mathcal{S}\mathrm{urj}_*$ with source $p(\underline{X})$ has a p-cocartesian lift $\bar{\alpha}$ in $\mathcal{O}^{\otimes}$ with source $\underline{X}$;
(2) for every $n \geq 0$, the p-cocartesian lifts $\bar{\rho}_i$ determine an equivalence of ∞-categories
$$\bar{\rho} : \mathcal{O}^{\otimes}_{\langle n\rangle} \simeq (\mathcal{O}^{\otimes}_{\langle 1\rangle})^n$$
where $\mathcal{O}^{\otimes}_{\langle n\rangle}$ denotes the fibre $p^{-1}(\langle n\rangle)$;
(3) for every pair of objects $\underline{X}, \underline{Y} \in \mathcal{O}^{\otimes}$ with $p(\underline{Y}) = \langle n\rangle$, the p-cocartesian lifts $\bar{\rho}_i : \underline{Y} \to Y_i$ determine an equivalence
$$\mathrm{Hom}_{\mathcal{O}^{\otimes}}(\underline{X}, \underline{Y}) \to \prod_{i=1}^{n} \mathrm{Hom}_{\mathcal{O}^{\otimes}}(\underline{X}, Y_i).$$

Since all ∞-operads appearing in this paper will be non-unital, we drop that adjective. We also commonly leave the map p implied and refer to *the* ∞*-operad* $\mathcal{O}^{\otimes}$. We write $\mathcal{O} = \mathcal{O}^{\otimes}_{\langle 1\rangle}$ and refer to this as the *underlying* ∞*-category* for the ∞-operad $\mathcal{O}^{\otimes}$.

Remark 4.2. An object $\underline{X} \in \mathcal{O}^{\otimes}$ with $p(\underline{X}) = S$ can be identified with a collection of objects of $\mathcal{O}$ indexed by S: a bijection $\alpha : S \cong \langle n\rangle$ induces a sequence of equivalences

$$\mathcal{O}^{\otimes}_S \xrightarrow[\simeq]{\bar{\alpha}} \mathcal{O}^{\otimes}_{\langle n\rangle} \simeq \mathcal{O}^n \simeq \prod_S \mathcal{O}.$$

Based on this observation, we will typically use a finite sequence of objects in $\mathcal{O}$ as a representative for an arbitrary object of $\mathcal{O}^{\otimes}$. For example, in (3) above, we identify the object $\underline{Y}$ with the sequence $(Y_1, \dots, Y_n)$.

Remark 4.3. An ∞-operad $\mathcal{O}^{\otimes}$ is an ∞-categorical version of a simplicial coloured operad whose colours are the objects of the underlying ∞-category $\mathcal{O}$. Given objects $X_1, \dots, X_n, Y \in \mathcal{O}$, for $n \geq 1$, we write

$$\mathrm{Hom}_{\mathcal{O}^{\otimes}}(X_1, \dots, X_n; Y) := \mathrm{Hom}_{\mathcal{O}^{\otimes}}((X_1, \dots, X_n), Y)_{f:\langle n\rangle \to \langle 1\rangle}$$

for the fibre of the morphism space in $\mathcal{O}^{\otimes}$ over the unique active morphism $f : \langle n\rangle \to \langle 1\rangle$ in $\mathcal{S}\mathrm{urj}_*$. We call these the *multi-morphism spaces* of the ∞-operad $\mathcal{O}^{\otimes}$. These spaces admit composition maps that are associative up to homotopy and through which we can view $\mathcal{O}^{\otimes}$ as a version of a coloured operad of simplicial sets. The definition of ∞-operad ensures that all mapping spaces of $\mathcal{O}^{\otimes}$ are determined by the multi-morphism spaces described here.

Definition 4.4. Given ∞-operads $p_1 : \mathcal{O}_1^{\otimes} \to \mathcal{S}\mathrm{urj}_*$ and $p_2 : \mathcal{O}_2^{\otimes} \to \mathcal{S}\mathrm{urj}_*$, a *map of ∞-operads* $g : \mathcal{O}_1^{\otimes} \to \mathcal{O}_2^{\otimes}$ is a functor g such that $p_2 \circ g = p_1$, and that sends p_1-cocartesian lifts in $\mathcal{O}_1^{\otimes}$ of inert maps in $\mathcal{S}\mathrm{urj}_*$ to p_2-cocartesian lifts in $\mathcal{O}_2^{\otimes}$. An *equivalence* of ∞-operads is a map of ∞-operads that is an equivalence on the underlying ∞-categories.

Definition 4.5. Let $p : \mathcal{O}^{\otimes} \to \mathcal{S}\mathrm{urj}_*$ be an ∞-operad, and let $\mathcal{O}'$ be a full subcategory of the underlying ∞-category $\mathcal{O}$. Then we let $\mathcal{O}'^{\otimes}$ be the full subcategory of $\mathcal{O}^{\otimes}$ whose objects are those equivalent (via the identifications of Remark 4.2) to sequences $(X_1, \dots, X_n)$ where $X_1, \dots, X_n \in \mathcal{O}'$. Then the restriction of p to $\mathcal{O}'^{\otimes}$ is also an ∞-operad, and the inclusion $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ is a map of ∞-operads. We refer to $\mathcal{O}'^{\otimes}$ as *the suboperad of $\mathcal{O}^{\otimes}$ generated by $\mathcal{O}'$*.

Definition 4.6. A *non-unital symmetric monoidal ∞-category* is a non-unital ∞-operad $p : \mathcal{C}^{\otimes} \to \mathcal{S}\mathrm{urj}_*$ such that p is a cocartesian fibration. This condition implies that for $X_1, \dots, X_n, Y \in \mathcal{C}$, we have

$$\mathrm{Hom}_{\mathcal{C}^{\otimes}}(X_1, \dots, X_n; Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X_1 \otimes \cdots \otimes X_n, Y)$$

for some object $X_1 \otimes \cdots \otimes X_n$ that depends functorially on $X_1, \dots, X_n$, and such that the operation $\otimes$ is associative and commutative up to higher coherent homotopies. This definition mimics the way in which a symmetric monoidal category can be viewed as a special kind of coloured operad.

A map of ∞-operads $g : \mathcal{C}_1^{\otimes} \to \mathcal{C}_2^{\otimes}$ between non-unital symmetric monoidal ∞-categories is *symmetric monoidal* if it takes cocartesian morphisms in $\mathcal{C}_1^{\otimes}$ to cocartesian morphisms in $\mathcal{C}_2^{\otimes}$.

The ∞-operads we study in this paper are stable in the following sense.

Definition 4.7. An ∞-operad $\mathcal{O}^{\otimes}$ is *stable* if the underlying ∞-category $\mathcal{O}$ is stable (in the sense of [18, 1.1.1.9]) and, for each $n \geq 1$, the functor

$$(\mathcal{O}^{op})^n \times \mathcal{O} \to \mathcal{T}op; \quad (X_1, \dots, X_n, Y) \mapsto \mathrm{Hom}_{\mathcal{O}^{\otimes}}(X_1, \dots, X_n; Y)$$

preserves finite limits in each variable. In that case, those functors are linear in each variable and so factor via corresponding spectrum-valued functors which we denote

$$\mathrm{Map}_{\mathcal{O}^{\otimes}}(X_1, \dots, X_n; Y).$$

We refer to these as the *multi-morphism spectra* of the stable ∞-operad $\mathcal{O}^{\otimes}$.

Example 4.8. A symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is stable if and only if $\mathcal{C}$ is stable and the monoidal product $\otimes$ is exact in each variable. In that case we have

$$\mathrm{Map}_{\mathcal{C}^{\otimes}}(X_1, \dots, X_n, Y) \simeq \mathrm{Map}_{\mathcal{C}}(X_1 \otimes \cdots \otimes X_n, Y).$$

Example 4.9. There is a non-unital symmetric monoidal ∞-category $\mathcal{S}p^{\wedge} \to \mathcal{S}\mathrm{urj}_*$ whose underlying ∞-category is $\mathcal{S}p$ and whose monoidal structure represents the ordinary smash product of spectra. See [18, 4.8.2] for the unital version which is a cocartesian fibration $\mathcal{S}p^{\wedge}_{\mathsf{u}} \to \mathcal{F}\mathrm{in}_*$. The required non-unital ∞-operad is given by pulling back this fibration along the inclusion $\mathcal{S}\mathrm{urj}_* \to \mathcal{F}\mathrm{in}_*$.

Example 4.10. Let $\mathcal{O}^{\otimes}$ be a stable ∞-operad with $\mathcal{O}$ equivalent to the ∞-category of finite spectra. Then the multi-morphism spectra for $\mathcal{O}^{\otimes}$ are determined by their values on the sphere spectrum. In particular, the data of $\mathcal{O}^{\otimes}$ are determined by the symmetric sequence of spectra

$$\mathbf{O}(n) := \mathrm{Map}_{\mathcal{O}^{\otimes}}(\underbrace{S^0, \dots, S^0}_{n}; S^0)$$

together with appropriate composition maps (that are associative up to higher coherent homotopies). In this way, $\mathcal{O}^{\otimes}$ can be viewed as the ∞-categorical version of an ordinary monochromatic operad of spectra.

We now turn to the main subject of this section, and we start with the construction of a symmetric monoidal ∞-category that represents the objectwise smash product of functors in $\mathcal{F}_{\mathcal{C}}$.

Construction 4.11. Consider the pullback of simplicial sets of the form

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{C}, \mathcal{S}p)^{\wedge} & \longrightarrow & \mathrm{Fun}(\mathcal{C}, \mathcal{S}p^{\wedge}) \\ {\scriptstyle p_{\mathcal{C}}}\downarrow & & \downarrow \\ \mathcal{S}\mathrm{urj}_* & \longrightarrow & \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{urj}_*) \end{array}$$

where the right-hand map is induced by the cocartesian fibration $\mathcal{S}p^{\wedge} \to \mathcal{S}\mathrm{urj}_*$ and the bottom map sends a finite pointed set to the constant functor with that value. The induced map $p_{\mathcal{C}}$ is then also a cocartesian fibration of ∞-operads, with fibres

$$\mathrm{Fun}(\mathcal{C}, \mathcal{S}p)^{\wedge}_{\langle n \rangle} \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{S}p^{\wedge}_{\langle n \rangle}).$$

Thus $p_{\mathcal{C}}$ is a (non-unital) symmetric monoidal ∞-category with underlying ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{S}p)$ and monoidal product given by the objectwise smash product of functors. (See [18, Remark 2.1.3.4].)

Definition 4.12. Let $\mathcal{F}^{\wedge}_{\mathcal{C}} \to \mathcal{S}\mathrm{urj}_*$ denote the restriction of the symmetric monoidal ∞-category $p_{\mathcal{C}}$ of Construction 4.11 to the full subcategory generated by those functors $\mathcal{C} \to \mathcal{S}p$ that are reduced and preserve filtered colimits. Since this collection of functors is closed under the objectwise smash product, $\mathcal{F}^{\wedge}_{\mathcal{C}}$ is also a non-unital symmetric monoidal ∞-category. (Note that we are forced to deal with a non-unital symmetric monoidal structure here because the unit object for the objectwise smash product is not a reduced functor.)

Our next goal is to describe a symmetric monoidal ∞-category that represents Day convolution. For this, we use the following construction of Glasman [11].

Construction 4.13. Let $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ be non-unital symmetric monoidal ∞-categories such that $\mathcal{C}$ is small, $\mathcal{D}$ admits all small colimits, and the monoidal structure on $\mathcal{D}$ preserves colimits in each variable. Then there is a non-unital symmetric monoidal ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes}$ that represents the Day convolution of functors $\mathcal{C} \to \mathcal{D}$. Glasman's construction [11, 2.8] is explicitly for ordinary (i.e. not non-unital) symmetric monoidal ∞-categories, but an almost identical construction yields a non-unital version. In Appendix B we address a fibrewise version of that construction which is needed for our work on bimodules over ∞-operads. The construction we need here is a special case of that fibrewise version (with $S = *$).

We would like to apply 4.13 to functors $\mathcal{F}_{\mathcal{C}} \to \mathcal{S}p$, but since $\mathcal{F}_{\mathcal{C}}$ is not small, we cannot do this directly. However, $\mathcal{F}_{\mathcal{C}}$ is a compactly-generated ∞-category, i.e. is generated under filtered colimits by the small subcategory $\mathcal{F}_{\mathcal{C}}^{\omega}$ of compact objects.

Lemma 4.14. *Let $\mathcal{C}$ be a pointed compactly-generated ∞-category. Then the ∞-category $\mathcal{F}_{\mathcal{C}}$ is compactly-generated and the subcategory $\mathcal{F}_{\mathcal{C}}^{\omega}$ of compact objects is closed under the objectwise smash product of functors.*

Proof. Let $\mathcal{R} \subseteq \mathcal{F}_{\mathcal{C}}$ be the full subcategory generated by the *representable* functors, i.e. those of the form $R_x := \Sigma^{\infty} \mathrm{Hom}_{\mathcal{C}}(x, -)$ for some compact object $x \in \mathcal{C}$. It is a standard consequence of the Yoneda Lemma 2.10, and the fact that equivalences in $\mathcal{F}_{\mathcal{C}}$ are detected objectwise on compact objects in $\mathcal{C}$, that an arbitrary $F \in \mathcal{F}_{\mathcal{C}}$ is the colimit of the canonical diagram

$$F \simeq \operatorname*{colim}_{R_x \to F} R_x,$$

indexed by the overcategory $\mathcal{R}_{/F}$. It follows, by the argument of [17, 5.3.4.17], that an arbitrary F is a filtered colimit of finite colimits of diagrams in $\mathcal{R}$, and therefore that the compact objects in $\mathcal{F}_{\mathcal{C}}$ are the retracts of those finite colimits. In particular, $\mathcal{F}_{\mathcal{C}}$ is compactly-generated.

Finally, from Lemma 2.11 it follows that the objectwise smash product of two representable functors is compact, since the cross-effect construction commutes with filtered colimits. Thus, the objectwise smash product of any two compact functors is compact. □

Definition 4.15. Let $(\mathcal{F}_{\mathcal{C}}^{\omega})^{\wedge} \to \mathcal{S}\mathrm{urj}_*$ be the suboperad of the symmetric monoidal ∞-category $p_{\mathcal{C}}$ of Definition 4.12 generated by the compact objects in $\mathcal{F}_{\mathcal{C}}$. By [18, 2.2.1.1], this suboperad is an essentially small stable symmetric monoidal ∞-category.

Definition 4.16. Applying Construction 4.13 to the symmetric monoidal ∞-category of the previous paragraph, we get a new stable non-unital symmetric monoidal ∞-category

$$q_{\mathcal{C}} : \mathrm{Fun}(\mathcal{F}_{\mathcal{C}}^{\omega}, \mathcal{S}p)^{\otimes} \to \mathcal{S}\mathrm{urj}_*.$$

To proceed to the definition of the ∞-operad $\mathbb{I}_{\mathcal{C}}^{\otimes}$, we need one more general construction.

Construction 4.17. Let $q : \mathcal{E}^{\otimes} \to \mathcal{S}\mathrm{urj}_*$ be a non-unital symmetric monoidal ∞-category. Then Barwick, Glasman and Nardin [4, 3.6] define another non-unital symmetric monoidal ∞-category $q^{\vee,op} : \mathcal{E}^{op,\otimes} \to \mathcal{S}\mathrm{urj}_*$ that represents the induced symmetric monoidal structure on the opposite ∞-category of $\mathcal{E}$. Note that when $\mathcal{E}^{\otimes}$ is stable, so is $\mathcal{E}^{op,\otimes}$.

Definition 4.18. Applying 4.17 to the non-unital symmetric monoidal ∞-category $q_{\mathcal{C}}$ of Definition 4.16, there is a stable non-unital symmetric monoidal ∞-category

$$q_{\mathcal{C}}^{\vee,op} : \mathrm{Fun}(\mathcal{F}_{\mathcal{C}}^{\omega}, \mathcal{S}p)^{op,\otimes} \to \mathcal{S}\mathrm{urj}_*$$

that represents the monoidal structure corresponding to Day convolution on the opposite ∞-category of the category of functors $\mathcal{F}_{\mathcal{C}}^{\omega} \to \mathcal{S}p$. Note that the multi-morphism spectra in $\mathrm{Fun}(\mathcal{F}_{\mathcal{C}}^{\omega}, \mathcal{S}p)^{op,\otimes}$ are given by the mapping spectra

$$\mathrm{Nat}_{\mathcal{F}_{\mathcal{C}}^{\omega}}(\mathsf{A}, \mathsf{B}_1 \otimes \cdots \otimes \mathsf{B}_n)$$

where $\otimes$ denotes the Day convolution of functors $\mathcal{F}_{\mathcal{C}}^{\omega} \to \mathcal{S}p$. Comparing with Remark 2.7, this observation motivates the following definition, which is the central construction of this paper.

Definition 4.19. Let $\mathcal{C}$ be a pointed compactly-generated ∞-category. Then let $\mathbb{I}_{\mathcal{C}}^{\otimes}$ be the suboperad of the symmetric monoidal ∞-category $\mathrm{Fun}(\mathcal{F}_{\mathcal{C}}^{\omega}, \mathcal{S}p)^{op,\otimes}$ generated by those objects of the form

$$\partial_1(-)(X) : \mathcal{F}_{\mathcal{C}}^{\omega} \to \mathcal{S}p$$

for $X \in \mathcal{S}p(\mathcal{C})$. We will usually denote the object $\partial_1(-)(X)$ of the underlying ∞-category $\mathbb{I}_{\mathcal{C}}$ simply by X.

Proposition 4.20. *The ∞-operad $\mathbb{I}_{\mathcal{C}}^{\otimes}$ is stable and has multi-morphism spectra*

$$\mathrm{Map}_{\mathbb{I}_{\mathcal{C}}^{\otimes}}(X_1, \dots, X_n; Y) \simeq \partial_n I_{\mathcal{C}}(X_1, \dots, X_n; Y).$$

In particular, the underlying ∞-category of $\mathbb{I}_{\mathcal{C}}^{\otimes}$ is equivalent to $\mathcal{S}p(\mathcal{C})^{op}$.

Proof. Since $\mathbb{I}_{\mathcal{C}}^{\otimes}$ is a full subcategory of a stable symmetric monoidal ∞-category, it has multi-morphism spectra given by

$$\mathrm{Map}_{\mathbb{I}_{\mathcal{C}}^{\otimes}}(X_1, \dots, X_n; Y) \simeq \mathrm{Nat}_{\mathcal{F}_{\mathcal{C}}^{\omega}}(\partial_1(-)(Y), \partial_1(-)(X_1) \otimes \cdots \otimes \partial_1(-)(X_n)).$$

Note that the tensor symbol on the right-hand side here denotes Day convolution for functors $\mathcal{F}_{\mathcal{C}}^{\omega} \to \mathcal{S}p$, rather than $\mathcal{F}_{\mathcal{C}} \to \mathcal{S}p$, so we cannot directly apply Theorem 2.4. However, the functors $\partial_1(-)(X_i)$ and $\partial_n(-)(X_1, \dots, X_n)$ all preserve filtered colimits, so are equivalent to the left Kan extensions of their restrictions to $\mathcal{F}_{\mathcal{C}}^{\omega} \subseteq \mathcal{F}_{\mathcal{C}}$ (by [17, 5.3.5.8(2)]). It follows that the Day convolution calculated in the subcategory $\mathcal{F}_{\mathcal{C}}^{\omega}$ is equivalent to that calculated over $\mathcal{F}_{\mathcal{C}}$. We thus have, by Theorem 2.4,

$$\mathrm{Map}_{\mathbb{I}_{\mathcal{C}}^{\otimes}}(X_1, \dots, X_n; Y) \simeq \mathrm{Nat}_{\mathcal{F}_{\mathcal{C}}^{\omega}}(\partial_1(-)(Y), \partial_n(-)(X_1, \dots, X_n)).$$

Since $\partial_1(-)(Y)$ also preserves filtered colimits, a similar argument implies that in fact

$$\mathrm{Map}_{\mathbb{I}_{\mathcal{C}}^{\otimes}}(X_1, \dots, X_n; Y) \simeq \mathrm{Nat}_{\mathcal{F}_{\mathcal{C}}}(\partial_1(-)(Y), \partial_n(-)(X_1, \dots, X_n))$$

which yields the desired formula by Corollary 3.7.

In particular, the underlying ∞-category of $\mathbb{I}_{\mathcal{C}}^{\otimes}$ has mapping spectra

$$\mathrm{Map}_{\mathbb{I}_{\mathcal{C}}}(X,Y) \simeq \partial_1(I_{\mathcal{C}})(X;Y) \simeq \mathrm{Map}_{\mathcal{S}p(\mathcal{C})}(Y,X)$$

so is equivalent to $\mathcal{S}p(\mathcal{C})^{op}$. It also now follows that the ∞-operad $\mathbb{I}_{\mathcal{C}}^{\otimes}$ is stable. □

Remark 4.21. It will sometimes be convenient to restrict $\mathbb{I}_{\mathcal{C}}^{\otimes}$ to the *small* ∞-operad whose underlying objects are the functors $\partial_1(-)(X)$ for *compact* objects X in $\mathcal{S}p(\mathcal{C})$. Since $\mathcal{S}p(\mathcal{C})$ is compactly-generated by [18, 1.4.3.7], those compact objects generate $\mathcal{S}p(\mathcal{C})$ under filtered colimits. Moreover, the functor $\partial_n I_{\mathcal{C}}$ preserves filtered colimits in each of its variables, and so the previous proposition shows that the ∞-operad $\mathbb{I}_{\mathcal{C}}^{\otimes}$ is determined by its restriction to these compact objects in a canonical way.

Remark 4.22. The ∞-operad $\mathbb{I}_{\mathcal{C}}^{\otimes}$ satisfies the additional property of being *corepresentable* in the sense of [18, 6.2.4.3], that is, the structure map $\mathbb{I}_{\mathcal{C}}^{\otimes} \to \mathcal{S}\mathrm{urj}_*$ is a locally cocartesian fibration. In other language, we can think of $\mathbb{I}_{\mathcal{C}}^{\otimes}$ as encoding an *oplax normal* symmetric monoidal structure on the underlying ∞-category $\mathcal{S}p(\mathcal{C})^{op}$ or, equivalently, a *lax normal* symmetric monoidal structure on $\mathcal{S}p(\mathcal{C})$ in the sense of [10].

More explicitly, this lax monoidal structure consists of the functors

$$\mathbf{\Delta}_n(I_{\mathcal{C}}) : \mathcal{S}p(\mathcal{C})^n \to \mathcal{S}p(\mathcal{C})$$

associated to the layers in the Taylor tower of $I_{\mathcal{C}}$, together with suitably compatible natural transformations

$$\mathbf{\Delta}_n(I_{\mathcal{C}})(X_1,\dots,\mathbf{\Delta}_r(I_{\mathcal{C}})(Y_1,\dots,Y_r),\dots,X_n) \to \mathbf{\Delta}_{n+r-1}(I_{\mathcal{C}})(X_1,\dots,Y_1,\dots,X_n).$$

Such a structure is also referred to sometimes as a *functor-operad* [21, 2.7].

Lemma 4.23. *The ∞-operad map $\mathbb{I}_{\mathcal{C}}^{\otimes} \to \mathcal{S}\mathrm{urj}_*$ is a locally cocartesian fibration.*

Proof. This follows from [18, Remark 6.2.4.5] and the natural equivalences

$$\mathrm{Map}_{\mathbb{I}_{\mathcal{C}}^{\otimes}}(X_1,\dots,X_n;Y) \simeq \partial_n(I_{\mathcal{C}})(X_1,\dots,X_n;Y) \simeq \mathrm{Map}_{\mathcal{S}p(\mathcal{C})}(Y,\mathbf{\Delta}_n(I_{\mathcal{C}})(X_1,\dots,X_n))$$

of 4.20 and 1.1. □

We have therefore proved the following result, which verifies Conjecture 6.3.0.13 of [18].

Proposition 4.24. *Let $\mathcal{C}$ be a pointed compactly-generated ∞-category. Then there is a stable corepresentable ∞-operad $\mathbb{I}_{\mathcal{C}}^{\otimes}$ with underlying ∞-category $\mathcal{S}p(\mathcal{C})^{op}$ whose corresponding lax monoidal structure consists of the symmetric multilinear functors*

$$\mathbf{\Delta}_n(I_{\mathcal{C}}) : \mathcal{S}p(\mathcal{C})^n \to \mathcal{S}p(\mathcal{C})$$

associated to the Taylor tower of $I_{\mathcal{C}}$.

Remark 4.25. One would expect there to be a close relationship between the ∞-operad $\mathbb{I}_{\mathcal{C}}^{\otimes}$ constructed here and Lurie's ∞-operad $\mathcal{S}p(\mathcal{C})^{\otimes}$ of [18, 6.3.0.14]. As described in [18, 6.3.0.17], we expect these two ∞-operads to be Koszul dual, but as far as we know a theory of Koszul duality for (stable) ∞-operads has not yet been sufficiently developed to allow this conjecture to be verified.

5. Stable algebras over infinity-operads

We now turn to our main example: the case where $\mathcal{C}$ is an ∞-category of algebras over a (non-unital) stable ∞-operad. In particular, this covers the 'classical' case of algebras over a (reduced) operad of spectra: for example, the reader may have in mind (non-unital) A_∞- or E_∞-ring spectra.

Let $\mathcal{O}^{\otimes}$ be a stable non-unital ∞-operad, and let $\mathsf{Alg}_{\mathcal{O}^{\otimes}}$ be the category of stable $\mathcal{O}^{\otimes}$-algebras defined in 5.1 below. It is a well-known slogan in Goodwillie calculus that the derivatives of the identity functor on $\mathsf{Alg}_{\mathcal{O}^{\otimes}}$ should be equivalent to $\mathcal{O}^{\otimes}$ itself. For example, in the monochromatic case a model for the Taylor tower for the identity on $\mathsf{Alg}_{\mathcal{O}^{\otimes}}$ is constructed by Pereira in [20], where the derivatives can be read off directly as the terms of the operad $\mathcal{O}^{\otimes}$. This tower was also studied by Harper and Hess in [16]. The goal of this section is to improve that slogan to a version that takes the operad structures into account.

We are therefore interested in comparing the ∞-operad $\mathbb{I}^{\otimes}_{\mathsf{Alg}_{\mathcal{O}^{\otimes}}}$, given by applying Definition 4.19 to $\mathsf{Alg}_{\mathcal{O}^{\otimes}}$, with the ∞-operad $\mathcal{O}^{\otimes}$ itself. We will see, however, that the underlying ∞-categories of these two ∞-operads are not equivalent, thus precluding an actual equivalence of ∞-operads between them. Instead, we will identify $\mathbb{I}^{\otimes}_{\mathsf{Alg}_{\mathcal{O}^{\otimes}}}$ with an ∞-operad $\mathrm{Pro}(\mathcal{O})^{\otimes}$ whose underlying ∞-category is that of pro-objects in $\mathcal{O}$. However, there is a fully-faithful embedding of $\mathcal{O}^{\otimes}$ into $\mathrm{Pro}(\mathcal{O})^{\otimes}$, so we will be able to identify $\mathcal{O}^{\otimes}$ with a full suboperad of $\mathbb{I}^{\otimes}_{\mathsf{Alg}_{\mathcal{O}^{\otimes}}}$.

Moreover, we will show that the inclusion of this suboperad induces an equivalence

$$\mathsf{Alg}_{\mathbb{I}^{\otimes}_{\mathsf{Alg}_{\mathcal{O}^{\otimes}}}} \xrightarrow{\sim} \mathsf{Alg}_{\mathcal{O}^{\otimes}}$$

which we take to mean that $\mathbb{I}^{\otimes}_{\mathsf{Alg}_{\mathcal{O}^{\otimes}}}$ is *Morita*-equivalent to $\mathcal{O}^{\otimes}$. This is our most precise version of the slogan mentioned at the beginning of this section.

Definition 5.1. Let $\mathcal{O}^{\otimes}$ be a small stable non-unital ∞-operad. A *stable* $\mathcal{O}^{\otimes}$*-algebra* is a map of (non-unital) ∞-operads

$$X : \mathcal{O}^{\otimes} \to \mathcal{S}p^{\wedge}$$

such that the underlying functor $X : \mathcal{O} \to \mathcal{S}p$ is exact. Let $\mathsf{Alg}_{\mathcal{O}^{\otimes}}$ denote the ∞-category of stable $\mathcal{O}^{\otimes}$-algebras, a full subcategory of $\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{S}p^{\wedge})$.

Example 5.2. Let $\mathcal{O}^{\otimes}$ be an ∞-operad whose underlying ∞-category is equivalent to $\mathcal{S}p^{\omega}$, the ∞-category of finite spectra. Then, as described in Example 4.10, $\mathcal{O}^{\otimes}$ corresponds to an ordinary reduced monochromatic operad $\mathbf{O}$ in the symmetric monoidal model category of spectra. The ∞-category $\mathsf{Alg}_{\mathcal{O}^{\otimes}}$ is equivalent to that of fibrant-cofibrant objects in the projective model structure on the category of (non-unital) $\mathbf{O}$-algebras in symmetric spectra. For example, if $\mathcal{O}^{\otimes} = (\mathcal{S}p^{\omega})^{\wedge}$ is the commutative stable ∞-operad, then $\mathsf{Alg}_{\mathcal{O}^{\otimes}}$ is equivalent to the ∞-category of (non-unital) E_∞-ring spectra.

Lemma 5.3. *For a small stable non-unital ∞-operad $\mathcal{O}^{\otimes}$, the ∞-category $\mathsf{Alg}_{\mathcal{O}^{\otimes}}$ is pointed and compactly-generated.*

Proof. The ∞-category $\mathsf{Alg}_{\mathcal{O}^{\otimes}}$ is a full subcategory of the ∞-category of all (unstable) $\mathcal{O}^{\otimes}$-algebras in the symmetric monoidal ∞-category $\mathcal{S}p$ (that is, the ∞-operad maps $\mathcal{O}^{\otimes} \to \mathcal{S}p^{\wedge}$ with no restriction on exactness of the underlying functor). Since that ∞-category of all algebras is compactly-generated by [18, 5.3.1.17], it is sufficient to show that this full subcategory is closed under limits and filtered colimits.

However, since limits and filtered colimits of algebras are computed objectwise, and both limits and filtered colimits commute with finite limits, these constructions preserve exactness of the underlying functor. □

We now wish to calculate the derivatives of the identity functor on $\mathsf{Alg}_{\mathcal{O}^{\otimes}}$, and the first step is to identify the stabilization of this ∞-category of algebras. This is a generalization of a result of Basterra and Mandell [5].

Proposition 5.4. *Let $\mathcal{O}^{\otimes}$ be a small stable non-unital ∞-operad. Then there is an equivalence of ∞-categories*

$$\mathcal{S}p(\mathsf{Alg}_{\mathcal{O}^{\otimes}}) \simeq \mathrm{Fun}^{\mathsf{ex}}(\mathcal{O}, \mathcal{S}p)$$

where $\mathcal{O}$ is the underlying ∞-category of $\mathcal{O}^{\otimes}$, and the right-hand side is the ∞-category of exact functors $\mathcal{O} \to \mathcal{S}p$.

Proof. With $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ as in the proof of Lemma 5.3, [18, 7.3.4.7] implies that the forgetful functor induces an equivalence

$$\mathcal{S}p(\mathsf{Alg}_{/\mathcal{O}}(\mathcal{C})) \simeq \mathrm{Fun}(\mathcal{O}, \mathcal{S}p)$$

which implies the desired result on restriction to exact functors. □

Proposition 5.4 implies that $\mathbb{I}^{\otimes}_{\mathsf{Alg}_{\mathcal{O}^{\otimes}}}$ is an ∞-operad whose underlying ∞-category is equivalent to $\mathrm{Fun}^{\mathsf{ex}}(\mathcal{O}, \mathcal{S}p)^{op}$. This ∞-category, however, has another interpretation.

Definition 5.5. Let $\mathcal{C}$ be a small ∞-category. Then the ∞-category of *pro-objects* on $\mathcal{C}$ is the full subcategory

$$\mathrm{Pro}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{T}op)^{op}$$

consisting of those functors $\mathcal{C} \to \mathcal{T}op$ that preserve finite limits.

Lemma 5.6. *If $\mathcal{C}$ is a small stable ∞-category, then composition with Ω^{∞} determines an equivalence*

$$\mathrm{Fun}^{\mathsf{ex}}(\mathcal{C}, \mathcal{S}p)^{op} \xrightarrow{\sim} \mathrm{Pro}(\mathcal{C}).$$

In particular, for a small stable ∞-operad $\mathcal{O}^{\otimes}$, there is an equivalence

$$\mathcal{S}p(\mathsf{Alg}_{\mathcal{O}^{\otimes}})^{op} \simeq \mathrm{Pro}(\mathcal{O}).$$

Example 5.7. Let $\mathcal{O}^{\otimes}$ be a stable non-unital ∞-operad whose underlying ∞-category is that of finite spectra, corresponding to an ordinary operad $\mathbf{O}$ of spectra as in Example 4.10. Then $\mathcal{S}p(\mathsf{Alg}_{\mathcal{O}^{\otimes}}) \simeq \mathcal{S}p$, and $\mathbb{I}^{\otimes}_{\mathsf{Alg}_{\mathcal{O}^{\otimes}}}$ is a corepresentable stable non-unital ∞-operad whose underlying ∞-category is equivalent to $\mathrm{Pro}(\mathcal{S}p^{\omega}) \simeq \mathcal{S}p^{op}$. Such an object can also be identified with an ordinary operad $\mathbf{O}$.

The goal of the rest of this section is to identify the ∞-operad structure on $\mathrm{Pro}(\mathcal{O})$ that corresponds to the derivatives of the identity on $\mathsf{Alg}_{\mathcal{O}^{\otimes}}$. Note first that when $\mathcal{O}^{\otimes}$ is a symmetric monoidal ∞-category we already have a candidate for this structure given by the (opposite of the) Day convolution. To extend that construction to an arbitrary $\mathcal{O}^{\otimes}$ we make use of the symmetric monoidal envelope construction of [18, 2.2.4].

Definition 5.8 (Lurie, [18, 2.2.4.1])**.** Let $\mathcal{O}^{\otimes} \to \mathcal{S}\mathrm{urj}_*$ be a small non-unital ∞-operad. The *symmetric monoidal envelope* of $\mathcal{O}^{\otimes}$ is the non-unital symmetric monoidal ∞-category

$$\mathcal{O}^{\otimes}_{act} \to \mathcal{S}\mathrm{urj}_*$$

where $\mathcal{O}^{\otimes}_{act} := \mathcal{O}^{\otimes} \times_{\mathcal{S}\mathrm{urj}_*} \mathrm{Act}(\mathcal{S}\mathrm{urj}_*)$ and $\mathrm{Act}(\mathcal{S}\mathrm{urj}_*)$ is the category of active morphisms in $\mathcal{S}\mathrm{urj}_*$ (objects are active morphisms and morphisms are commutative squares).

Note that the objects of the underlying ∞-category $\mathcal{O}_{act}$ can be identified with the objects of $\mathcal{O}^{\otimes}$, i.e. unordered finite sequences of objects in $\mathcal{O}$.

Remark 5.9. The symmetric monoidal ∞-category $\mathcal{O}^{\otimes}_{act}$ has the following universal property: for any symmetric monoidal ∞-category $\mathcal{D}^{\otimes}$, there is an equivalence between the ∞-category of ∞-operad maps $\mathcal{O}^{\otimes} \to \mathcal{D}^{\otimes}$ and that of symmetric monoidal functors $\mathcal{O}^{\otimes}_{act} \to \mathcal{D}^{\otimes}$.

Definition 5.10. We apply Constructions 4.13 and 4.17 to $\mathcal{O}_{act}$ to build a (non-unital) stable symmetric monoidal ∞-category

$$\mathrm{Fun}(\mathcal{O}_{act}, \mathcal{S}p)^{op,\otimes}$$

whose objects are the functors $\mathcal{O}_{act} \to \mathcal{S}p$, and with symmetric monoidal structure given by Day convolution.

Lemma 5.11. *There is a fully-faithful embedding of stable ∞-categories*

$$\mathrm{Pro}(\mathcal{O}) \simeq \mathrm{Fun}^{\mathsf{ex}}(\mathcal{O}, \mathcal{S}p)^{op} \xrightarrow{l} \mathrm{Fun}(\mathcal{O}_{act}, \mathcal{S}p)^{op}$$

where l is given by left Kan extension along the inclusion $i : \mathcal{O} \hookrightarrow \mathcal{O}_{act}$. The essential image of l consists of those functors $F : \mathcal{O}_{act} \to \mathcal{S}p$ such that

(1) *$F(I_1, \dots, I_n) \simeq *$ for any $I_1, \dots, I_n \in \mathcal{O}$ where $n > 1$;*
(2) *the restriction of F to $\mathcal{O} \subseteq \mathcal{O}_{act}$ is an exact functor $\mathcal{O} \to \mathcal{S}p$.*

Proof. It is sufficient to show that the left Kan extension along i determines a fully-faithful embedding

$$l : \mathrm{Fun}(\mathcal{O}, \mathcal{S}p) \to \mathrm{Fun}(\mathcal{O}_{act}, \mathcal{S}p)$$

whose essential image consists of those functors $F : \mathcal{O}_{act} \to \mathcal{S}p$ satisfying condition (1). First note that i is fully-faithful, so l is too. Then since the ∞-category $\mathcal{O}_{act}$ admits no morphisms of the form $I \to (I_1, \dots, I_n)$ where $n > 1$, the left Kan extension takes values as desired. □

Definition 5.12. Let $\mathrm{Pro}(\mathcal{O})^{\otimes}$ be the full suboperad of $\mathrm{Fun}(\mathcal{O}_{act}, \mathcal{S}p)^{op,\otimes}$ generated by those functors $F : \mathcal{O}_{act} \to \mathcal{S}p$ that satisfy conditions (1) and (2) of Lemma 5.11. This suboperad has underlying ∞-category equivalent (via the map l of 5.11) to $\mathrm{Pro}(\mathcal{O})$.

Remark 5.13. We can describe the ∞-operad structure on $\mathrm{Pro}(\mathcal{O})$ in the following way. Suppose we have exact functors $X_1, \dots, X_n, Y : \mathcal{O} \to \mathcal{S}p$. Then the multi-morphism spectrum

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{O})^{\otimes}}(X_1, \dots, X_n; Y)$$

is equivalent to the spectrum of natural transformations (of functors $\mathcal{O}_{act} \to \mathcal{S}p$) of the form

$$lY \to lX_1 \otimes \cdots \otimes lX_n$$

where $\otimes$ denotes the Day convolution of functors $\mathcal{O}_{act} \to \mathcal{S}p$. Equivalently, this is the spectrum of natural transformations (of functors $\mathcal{O} \to \mathcal{S}p$) of the form

$$Y \to r(lX_1 \otimes \cdots \otimes lX_n)$$

where r is restriction along $i : \mathcal{O} \hookrightarrow \mathcal{O}_{act}$. In other words, the ∞-operad structure on $\mathrm{Pro}(\mathcal{O})$ is corepresented by the construction

$$(X_1, \dots, X_n) \mapsto r(lX_1 \otimes \cdots \otimes lX_n)$$

which we can write, using coend notation for the Day convolution, as the functor

$$J \mapsto \mathrm{Map}_{\mathcal{O}^{\otimes}}(I_1, \dots, I_n; J) \wedge_{I_1, \dots, I_n \in \mathcal{O}} X_1(I_1) \wedge \ldots \wedge X_n(I_n).$$

The main result of this section is that the ∞-operad $\mathbb{I}^{\otimes}_{\mathsf{Alg}_{\mathcal{O}^{\otimes}}}$ is equivalent to $\mathrm{Pro}(\mathcal{O})^{\otimes}$. In order to make this comparison, we require the following construction.

Definition 5.14. Let $\mathcal{O}^{\otimes}$ be a small stable non-unital ∞-operad. Using Construction 4.11 we can produce a map of ∞-operads

$$\mathsf{ev} : \mathcal{O}^{\otimes} \to \mathcal{F}^{\wedge}_{\mathsf{Alg}_{\mathcal{O}^{\otimes}}} \subseteq \mathrm{Fun}(\mathsf{Alg}_{\mathcal{O}^{\otimes}}, \mathcal{S}p)^{\wedge}$$

given, on underlying ∞-categories, by

$$I \mapsto \mathsf{ev}_I$$

where $\mathsf{ev}_I : \mathsf{Alg}_{\mathcal{O}^{\otimes}} \to \mathcal{S}p$ is the functor that evaluates an $\mathcal{O}^{\otimes}$-algebra X at the object $I \in \mathcal{O}$. The map ev then extends canonically to a symmetric monoidal functor

$$\mathsf{ev} : \mathcal{O}^{\otimes}_{act} \to \mathcal{F}^{\bar{\wedge}}_{\mathsf{Alg}_{\mathcal{O}^{\otimes}}}$$

given by

$$(I_1, \dots, I_n) \mapsto \mathsf{ev}_{I_1} \wedge \ldots \wedge \mathsf{ev}_{I_n},$$

and restriction along this ev in turn determines a symmetric monoidal functor

$$\mathsf{ev}^* : \mathrm{Fun}(\mathcal{F}_{\mathsf{Alg}_{\mathcal{O}^{\otimes}}}, \mathcal{S}p)^{op,\otimes} \to \mathrm{Fun}(\mathcal{O}_{act}, \mathcal{S}p)^{op,\otimes}.$$

Remark 5.15. There is one technical wrinkle in the previous definition that we have to be careful with. Recall that the ∞-category $\mathcal{F}_{\mathsf{Alg}_{\mathcal{O}^\otimes}}$ is not small and so, in the construction of $\mathbb{I}_{\mathsf{Alg}_{\mathcal{O}^\otimes}}$ we replaced it with the small subcategory $\mathcal{F}^\omega_{\mathsf{Alg}_{\mathcal{O}^\otimes}}$ of compact objects. Since it is unclear whether the functors ev_I are compact, we will simply add them in the following way.

Let $\tilde{\mathcal{F}}^\omega_{\mathsf{Alg}_{\mathcal{O}^\otimes}}$ be the (essentially) small full subcategory of $\mathcal{F}_{\mathsf{Alg}_{\mathcal{O}^\otimes}}$ obtained from $\mathcal{F}^\omega_{\mathsf{Alg}_{\mathcal{O}^\otimes}}$ by adjoining the objects ev_I for all $I \in \mathcal{O}$, and then taking closure under the objectwise smash product. The functor ev takes values in this (essentially small) symmetric monoidal ∞-category, and we get an associated restriction functor

$$\mathsf{ev}^* : \mathrm{Fun}(\tilde{\mathcal{F}}^\omega_{\mathsf{Alg}_{\mathcal{O}^\otimes}}, \mathcal{S}p)^{op,\otimes} \to \mathrm{Fun}(\mathcal{O}_{act}, \mathcal{S}p)^{op,\otimes}.$$

Since $\tilde{\mathcal{F}}^\omega_{\mathsf{Alg}_{\mathcal{O}^\otimes}}$ generates $\mathcal{F}_{\mathsf{Alg}_{\mathcal{O}^\otimes}}$ under filtered colimits, the arguments of Proposition 4.20 imply that $\mathbb{I}^\otimes_{\mathsf{Alg}_{\mathcal{O}^\otimes}}$ can be identified with the suboperad of $\mathrm{Fun}(\tilde{\mathcal{F}}^\omega_{\mathsf{Alg}_{\mathcal{O}^\otimes}}, \mathcal{S}p)^{op,\otimes}$ generated by objects of the form $\partial_1(-)(X)$ for $X \in \mathcal{S}p(\mathsf{Alg}_{\mathcal{O}^\otimes})$.

The following is the main result of this section.

Theorem 5.16. *The composite*

$$\theta : \mathbb{I}^\otimes_{\mathsf{Alg}_{\mathcal{O}^\otimes}} \hookrightarrow \mathrm{Fun}(\tilde{\mathcal{F}}^\omega_{\mathsf{Alg}_{\mathcal{O}^\otimes}}, \mathcal{S}p)^{op,\otimes} \xrightarrow{\mathsf{ev}^*} \mathrm{Fun}(\mathcal{O}_{act}, \mathcal{S}p)^{op,\otimes}$$

is a fully faithful embedding of ∞-operads whose essential image is $\mathrm{Pro}(\mathcal{O})^\otimes$. *Thus we have an equivalence of ∞-operads*

$$\mathbb{I}^\otimes_{\mathsf{Alg}_{\mathcal{O}^\otimes}} \simeq \mathrm{Pro}(\mathcal{O})^\otimes.$$

Proof. We first show that the essential image is as claimed. An object of $\mathbb{I}_{\mathsf{Alg}_{\mathcal{O}^\otimes}}$ is a functor $\tilde{\mathcal{F}}^\omega_{\mathsf{Alg}_{\mathcal{O}^\otimes}} \to \mathcal{S}p$ of the form $\partial_1(-)(X)$ for $X \in \mathcal{S}p(\mathsf{Alg}_{\mathcal{O}^\otimes})$. We can easily calculate the first derivative of the evaluation functors $\mathsf{ev}_{I_1} \wedge \dots \wedge \mathsf{ev}_{I_n}$ for $I_1, \dots, I_n \in \mathcal{O}$. When $n > 1$, this first derivative is trivial, and when $n = 1$, we have

$$\partial_1(\mathsf{ev}_I)(X) \simeq X(I)$$

where we have identified $X \in \mathcal{S}p(\mathsf{Alg}_{\mathcal{O}^\otimes})$ with an exact functor $X : \mathcal{O} \to \mathcal{S}p$. This identifies the underlying functor of θ with the map l of Lemma 5.11.

It is now sufficient to show that θ induces equivalences on multi-mapping spectra. By Theorem 2.4, it is enough to show that restriction along ev induces equivalences

$$\begin{array}{c} \mathrm{Nat}_{\tilde{\mathcal{F}}^\omega_{\mathsf{Alg}_{\mathcal{O}^\otimes}}}(\partial_1(-)(Y), \partial_n(-)(X_1, \dots, X_n)) \\ \downarrow \mathsf{ev}^* \\ \mathrm{Nat}_{\mathcal{O}_{act}}(\partial_1(\mathsf{ev}_\bullet)(Y), \partial_n(\mathsf{ev}_\bullet)(X_1, \dots, X_n)) \end{array} \tag{5.17}$$

for all $X_1, \dots, X_n, Y \in \mathcal{S}p(\mathsf{Alg}_{\mathcal{O}^\otimes})$. This calculation is the real substance of our result on the derivatives of the identity for $\mathsf{Alg}_{\mathcal{O}^\otimes}$.

As mentioned above, $\partial_1(\mathsf{ev}_\bullet)(Y)$ is trivial when evaluated on objects $(I_1, \dots, I_n)$ in $\mathcal{O}_{act}$ with $n > 1$. It follows (since there are no maps in $\mathcal{O}_{act}$ of the form $J \to (I_1, \dots, I_n)$) that the target of (5.17) can be written as a natural transformation object over $\mathcal{O}$.

Then, precomposing with the equivalence of Theorem 3.1, we are reduced to showing that there is an equivalence

$$\mathrm{Map}_{\mathcal{S}p(\mathsf{Alg}_{\mathcal{O}^\otimes})}(Y, \mathbf{\Delta}_n I_{\mathsf{Alg}_{\mathcal{O}^\otimes}}(X_1, \dots, X_n)) \to \mathrm{Nat}_{\mathcal{O}}(\partial_1(\mathsf{ev}_\bullet)(Y), \partial_n(\mathsf{ev}_\bullet)(X_1, \dots, X_n))$$

given by the map described in Definition 3.4, i.e. induced by the maps

$$c : \mathbf{\Delta}_1(\mathsf{ev}_\bullet)(\mathbf{\Delta}_n I_{\mathsf{Alg}_{\mathcal{O}^\otimes}}(X_1, \dots, X_n)) \to \mathbf{\Delta}_n(\mathsf{ev}_\bullet)(X_1, \dots, X_n)$$

of (3.2).

Recalling that $\partial_1(\mathsf{ev}_\bullet)(Y) : \mathcal{O} \to \mathcal{S}p$ can be identified with Y itself under the equivalence of Proposition 5.4, we are reduced to showing that this map c is an equivalence. Looking at the construction of c in (3.3), however, we see that this follows immediately from the claim that

$$\mathsf{ev}_J \, \mathrm{cr}_n \, I_{\mathsf{Alg}_{\mathcal{O}^\otimes}} \simeq \mathrm{cr}_n(\mathsf{ev}_J)$$

which is a consequence of the fact that the evaluation map $\mathsf{ev}_J : \mathsf{Alg}_{\mathcal{O}^\otimes} \to \mathcal{S}p$ preserves limits. □

Having identified the ∞-operad $\mathbb{I}^\otimes_{\mathsf{Alg}_{\mathcal{O}^\otimes}}$ with $\mathrm{Pro}(\mathcal{O})^\otimes$, we now wish to relate this calculation to $\mathcal{O}^\otimes$ itself and justify the slogan mentioned at the beginning of this section. To do this, we will see that $\mathcal{O}^\otimes$ embeds fully-faithfully, via a stable Yoneda map, into $\mathrm{Pro}(\mathcal{O})^\otimes$, and hence is equivalent to a full suboperad of $\mathbb{I}^\otimes_{\mathsf{Alg}_{\mathcal{O}^\otimes}}$.

Glasman shows in [11, Sec. 3] that a symmetric monoidal ∞-category such as $\mathcal{O}^\otimes_{act}$ admits a multiplicative Yoneda embedding, that is, a fully-faithful symmetric monoidal map

$$Y_{act} : \mathcal{O}^\otimes_{act} \to \mathrm{Fun}(\mathcal{O}_{act}, \mathcal{T}op)^{op,\otimes}$$

which restricts to a fully-faithful embedding of ∞-operads

$$Y : \mathcal{O}^\otimes \to \mathrm{Fun}(\mathcal{O}_{act}, \mathcal{T}op)^{op,\otimes}.$$

whose underlying functor $Y : \mathcal{O} \to \mathrm{Fun}(\mathcal{O}_{act}, \mathcal{T}op)^{op}$ is the restriction of the ordinary Yoneda embedding, i.e. is given by

$$Y(I) : (J_1, \dots, J_n) \mapsto \mathrm{Hom}_{\mathcal{O}_{act}}(I, (J_1, \dots, J_n)) = \begin{cases} \mathrm{Hom}_{\mathcal{O}}(I, J_1) & \text{if } n = 1; \\ * & \text{otherwise.} \end{cases}$$

We can now easily see the following.

Lemma 5.18. *The map Y factors as*

$$\mathcal{O}^\otimes \xrightarrow{\bar{Y}} \mathrm{Pro}(\mathcal{O})^\otimes \xrightarrow{\Omega^\infty} \mathrm{Fun}(\mathcal{O}_{act}, \mathcal{T}op_*)^{op,\otimes}$$

where $\bar{Y}$ is a fully-faithful embedding of ∞-operads.

Proof. It is clear from the formula that the image of Y is contained in the full suboperad of $\mathrm{Fun}(\mathcal{O}_{act}, \mathcal{T}op)^{op,\otimes}$ whose objects are functors that are trivial on $(I_1, \dots, I_n) \in \mathcal{O}_{act}$ for $n > 1$, and restrict to an exact functor $\mathcal{O} \to \mathcal{T}op$. Since composition with Ω^∞ is an equivalence between the ∞-categories of exact functors from $\mathcal{O}$ to $\mathcal{S}p$, and from $\mathcal{O}$ to $\mathcal{T}op$, that suboperad is equivalent to $\mathrm{Pro}(\mathcal{O})^{\otimes}$. This gives the desired factorization, and since Y was fully faithful, so is $\bar{Y}$. □

In combination with Theorem 5.16, Lemma 5.18 then allows us to identify a suboperad of $\mathbb{I}^{\otimes}_{\mathsf{Alg}_{\mathcal{O}^\otimes}}$ with $\mathcal{O}^{\otimes}$ itself.

Theorem 5.19. *Let $\mathcal{O}^{\otimes}$ be a small stable non-unital ∞-operad. There is a suboperad of $\mathbb{I}^{\otimes}_{\mathsf{Alg}_{\mathcal{O}^\otimes}}$, generated by the representable pro-objects $\mathrm{Map}_{\mathcal{O}}(I, -) \in \mathcal{S}p(\mathsf{Alg}_{\mathcal{O}^\otimes})$, that is equivalent to $\mathcal{O}^{\otimes}$.*

In Remark 4.21 we noted that sometimes it makes sense to restrict to the small ∞-operad $\check{\mathbb{I}}^{\otimes}_{\mathcal{C}} \subseteq \mathbb{I}^{\otimes}_{\mathcal{C}}$ generated by the compact objects of $\mathcal{S}p(\mathcal{C})$. If we do that with $\mathcal{C} = \mathsf{Alg}_{\mathcal{O}^\otimes}$ then we get the small stable ∞-operad with underlying ∞-category

$$\check{\mathbb{I}}_{\mathsf{Alg}_{\mathcal{O}^\otimes}} \simeq \mathrm{Pro}(\mathcal{O})^{\omega} = (\mathrm{Ind}(\mathcal{O}^{op})^{\omega})^{op}$$

which, by [17, 5.4.2.4], is an idempotent-completion of $\mathcal{O}$. In particular, we have the following.

Corollary 5.20. *Let $\mathcal{O}^{\otimes}$ be a small stable ∞-operad with $\mathcal{O}$ idempotent-complete. Then there is an equivalence of ∞-operads*

$$\check{\mathbb{I}}^{\otimes}_{\mathsf{Alg}_{\mathcal{O}^\otimes}} \simeq \mathcal{O}^{\otimes}.$$

More generally, we can show that $\check{\mathbb{I}}^{\otimes}_{\mathsf{Alg}_{\mathcal{O}^\otimes}}$ is *Morita*-equivalent to $\mathcal{O}^{\otimes}$, i.e. that the inclusion induces an equivalence between the corresponding ∞-category of (stable) algebras.

Lemma 5.21. *Let $i : \mathcal{A}^{\otimes} \to \mathcal{B}^{\otimes}$ be a fully-faithful map of stable ∞-operads. Then the corresponding restriction map*

$$i^* : \mathsf{Alg}_{\mathcal{B}^\otimes} \to \mathsf{Alg}_{\mathcal{A}^\otimes}$$

has a fully-faithful left adjoint.

Proof. Limits and filtered colimits of (stable) algebras are calculate objectwise (by [18, 3.2.2.4] and [18, 3.2.3.1]), and so i^* preserves those (co)limits. Thus, by the Adjoint Functor Theorem [17, 5.5.2.9], the desired left adjoint $i_!$ exists. This left adjoint must be given by the same formula as for not-necessarily-stable algebras, described in [18, 3.1.3.1]: for an $\mathcal{A}^{\otimes}$-algebra X, and object $J \in \mathcal{B}$, we have

$$i_!(X)(J) = \operatorname*{colim}_{(\underline{I}_1, \dots, \underline{I}_n) \in (\mathcal{A}^{\otimes}_{act})_{/J}} X(\underline{I}_1) \wedge \dots \wedge X(\underline{I}_n).$$

Since $\mathcal{A}^{\otimes}_{act}$ is a full subcategory of $\mathcal{B}^{\otimes}_{act}$, it follows that the counit is an equivalence

$$i^* i_!(X) \xrightarrow{\sim} X$$

and so $i_!$ is fully-faithful. □

Theorem 5.22. *Precomposition with the inclusion of stable ∞-operads*

$$\bar{Y} : \mathcal{O}^{\otimes} \to \check{\mathbb{I}}^{\otimes}_{\mathsf{Alg}_{\mathcal{O}^{\otimes}}}$$

induces an equivalence of ∞-categories

$$\bar{Y}^* : \mathsf{Alg}_{\check{\mathbb{I}}^{\otimes}_{\mathsf{Alg}_{\mathcal{O}^{\otimes}}}} \xrightarrow{\sim} \mathsf{Alg}_{\mathcal{O}^{\otimes}}.$$

In other words, $\check{\mathbb{I}}^{\otimes}_{\mathsf{Alg}_{\mathcal{O}^{\otimes}}}$ is Morita-equivalent to $\mathcal{O}^{\otimes}$.

Proof. Given that $\bar{Y}$ is fully-faithful, Lemma 5.21 tells us that $\bar{Y}^*$ has a fully-faithful left adjoint $\bar{Y}_!$. It is then sufficient to show that $\bar{Y}^*$ is conservative. Since equivalences of algebras are detected on the underlying ∞-categories, this follows from the fact that $\check{\mathbb{I}}_{\mathsf{Alg}_{\mathcal{O}^{\otimes}}}$ is an idempotent-completion of $\mathcal{O}$ and [17, 5.1.4.9]. □

We conclude this section by noting that there is a natural comparison map between a pointed compactly-generated ∞-category $\mathcal{C}$ and the ∞-category of (stable) algebras over the ∞-operad $\mathbb{I}^{\otimes}_{\mathcal{C}}$.

Definition 5.23. Let $\mathsf{ev} : \mathcal{C} \to \mathrm{Fun}(\mathcal{F}^{\omega}_{\mathcal{C}}, \mathcal{S}p)$ be adjoint to the evaluation functor

$$\mathcal{C} \times \mathcal{F}^{\omega}_{\mathcal{C}} \to \mathcal{S}p; \quad (X, F) \mapsto F(X).$$

It follows from the universal property of the Day convolution in 2.2 that ev takes values in commutative monoid objects for the Day convolution symmetric monoidal structure on $\mathrm{Fun}(\mathcal{F}^{\omega}_{\mathcal{C}}, \mathcal{S}p)$.

The Yoneda embedding for the stable symmetric monoidal ∞-category $\mathrm{Fun}(\mathcal{F}^{\omega}_{\mathcal{C}}, \mathcal{S}p)^{\otimes}$ of Nikolaus [19, Sec. 6] determines a functor, there denoted j'_{St} from these commutative monoid objects to the ∞-category of stable algebras over $\mathrm{Fun}(\mathcal{F}^{\omega}_{\mathcal{C}}, \mathcal{S}p)^{\otimes,op}$. Composing and restricting to $\mathbb{I}^{\otimes}_{\mathcal{C}}$ we obtain the desired functor

$$\eta : \mathcal{C} \to \mathsf{Alg}_{\mathbb{I}^{\otimes}_{\mathcal{C}}}.$$

Remark 5.24. We conjecture that η is the unit of a 'quasi-adjunction' [15, Sec. 7] between the functors

$$\mathcal{C} \mapsto \check{\mathbb{I}}^{\otimes}_{\mathcal{C}}$$

and

$$\mathcal{O}^{\otimes} \mapsto \mathsf{Alg}_{\mathcal{O}^{\otimes}}$$

which, we conjecture, can be interpreted as relating certain $(\infty, 2)$-categories of pointed compactly-generated ∞-categories (and reduced finitary functors) and small stable non-unital ∞-operads (and bimodules between them). The counit of that quasi-adjunction would be made up of the equivalences of Theorem 5.22. To develop this theory properly is beyond the scope of this paper, though we will introduce bimodules between ∞-operads in the next section.

6. Bimodules over infinity-operads

Consider a functor $F : \mathcal{C} \to \mathcal{D}$ between pointed compactly-generated ∞-categories. We wish to show that the derivatives of F have the structure of a *bimodule* over the ∞-operads $\mathbb{I}^{\otimes}_{\mathcal{C}}$ and $\mathbb{I}^{\otimes}_{\mathcal{D}}$. Bimodules are studied by Lurie in [18, 3.1.2.1] under the guise of *correspondences*, or Δ^1*-families*, of ∞-operads. In order to state our conjectured chain rule, we need families of ∞-operads indexed by arbitrary simplicial sets, not just by Δ^1. The following is a non-unital version of [18, 2.3.2.10].

Definition 6.1. Let S be a simplicial set. An *S-family of non-unital ∞-operads* consists of a categorical fibration

$$p : \mathcal{M}^{\otimes} \to S \times \mathcal{S}\mathrm{urj}_*$$

with the following properties:

(1) the restriction $p_s : \mathcal{M}^{\otimes}_s \to \mathcal{S}\mathrm{urj}_*$ of p to each vertex $s \in S$ is an ∞-operad;
(2) for each object $(X_1, \dots, X_m) \in \mathcal{M}^{\otimes}_s$, each inert morphism $\alpha : \langle m \rangle \to \langle n \rangle$ in $\mathcal{S}\mathrm{urj}_*$ has a lift

$$\bar{\alpha} : (X_1, \dots, X_m) \to (X'_1, \dots, X'_n)$$

in $\mathcal{M}^{\otimes}_s$ such that $\bar{\alpha}$ is p-cocartesian (and not merely p_s-cocartesian);
(3) for each 1-simplex $f : s \to s'$ in S and each pair of objects $(X_1, \dots, X_m) \in \mathcal{M}_s$, and $(Y_1, \dots, Y_n) \in \mathcal{M}_{s'}$, the inert maps $(Y_1, \dots, Y_n) \to (Y_i)$ in $\mathcal{M}^{\otimes}_{s'}$ induce equivalences

$$\mathrm{Hom}_{\mathcal{M}^{\otimes}}((X_1, \dots, X_m), (Y_1, \dots, Y_n))_f \simeq \prod_{i=1}^{n} \mathrm{Hom}_{\mathcal{M}^{\otimes}}((X_1, \dots, X_m), (Y_i))_f$$

between the spaces of maps in $\mathcal{M}^{\otimes}$ that project via p to f.

We often leave the structure map p implied and refer to such a family of ∞-operads simply as $\mathcal{M}^{\otimes}$.

Definition 6.2. We say that an S-family of ∞-operads $\mathcal{M}^{\otimes} \to S \times \mathcal{S}\mathrm{urj}_*$ is *stable* if the ∞-operad $\mathcal{M}^{\otimes}_s$ is stable for each $s \in S$ and, for each n and each 1-simplex $f : s \to s'$ in S, the functor

$$\mathrm{Hom}_{\mathcal{M}^{\otimes}}(-, \dots, -; -)_f : (\mathcal{M}^{op}_s)^n \times \mathcal{M}_{s'} \to \mathcal{T}op$$

preserves finite limits in each variable, where the multi-morphism spaces are defined in the same manner as in Remark 4.3. In this case, as in Definition 4.7, we have corresponding multi-morphism spectra $\mathrm{Map}_{\mathcal{M}^{\otimes}}(X_1, \dots, X_n; Y)_f$.

Definition 6.3. Let $\mathcal{L}^{\otimes}$ and $\mathcal{R}^{\otimes}$ be stable non-unital ∞-operads. An $(\mathcal{L}^{\otimes}, \mathcal{R}^{\otimes})$-*bimodule* is a stable Δ^1-family of non-unital ∞-operads $p : \mathcal{M}^{\otimes} \to \Delta^1 \times \mathcal{S}\mathrm{urj}_*$ together with equivalences of ∞-operads

$$\mathcal{M}^{\otimes}_0 \simeq \mathcal{R}^{\otimes}, \quad \mathcal{M}^{\otimes}_1 \simeq \mathcal{L}^{\otimes}.$$

Remark 6.4. Let $\mathcal{M}^{\otimes}$ be an $(\mathcal{L}^{\otimes}, \mathcal{R}^{\otimes})$-bimodule. Then, for $X_1, \dots, X_n \in \mathcal{R}$ and $Y \in \mathcal{L}$, the multi-morphism spectra

$$\mathrm{Map}_{\mathcal{M}^{\otimes}}(X_1, \dots, X_n; Y)$$

have actions, on the left by the multi-morphism spectra of $\mathcal{L}^{\otimes}$, and on the right by the multi-morphism spectra of $\mathcal{R}^{\otimes}$, that form the structure of a bimodule over the coloured operads corresponding to $\mathcal{L}^{\otimes}$ and $\mathcal{R}^{\otimes}$, up to coherent homotopy.

Remark 6.5. We can think of a stable S-family of ∞-operads $\mathcal{M}^{\otimes} \to S \times \mathcal{S}\mathrm{urj}_*$ as a diagram of operads and bimodules over them indexed by the simplicial set S:

- for each vertex $s \in S$ we have an ∞-operad $\mathcal{M}^{\otimes}_s$;
- for each 1-simplex $f : s_0 \to s_1$ in S we have an $(\mathcal{M}^{\otimes}_{s_1}, \mathcal{M}^{\otimes}_{s_0})$-bimodule $\mathcal{M}^{\otimes}_f$
- for each 2-simplex

  $$\begin{array}{ccc} s_0 & \xrightarrow{h} & s_2 \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle g} & \\ & s_1 & \end{array}$$

  in S, we have a map of $(\mathcal{M}^{\otimes}_{s_2}, \mathcal{M}^{\otimes}_{s_0})$-bimodules

  $$\mathcal{M}^{\otimes}_g \circ_{\mathcal{M}^{\otimes}_{s_1}} \mathcal{M}^{\otimes}_f \to \mathcal{M}^{\otimes}_h$$

  where the left-hand side denotes a *relative composition product* of two bimodules over the ∞-operad $\mathcal{M}^{\otimes}_{s_1}$.

The last part of this description requires some explanation. Composition in the ∞-category $\mathcal{M}^{\otimes}$ determines maps of spectra

$$\begin{array}{c} \mathrm{Map}_{\mathcal{M}^{\otimes}}(Y_1, \dots, Y_n; Z)_g \wedge \mathrm{Map}_{\mathcal{M}^{\otimes}}(X_{11}, \dots; Y_1)_f \wedge \dots \wedge \mathrm{Map}_{\mathcal{M}^{\otimes}}(X_{n1}, \dots, X_{nk_n}; Y_n)_f \\ \Big\downarrow \\ \mathrm{Map}_{\mathcal{M}^{\otimes}}(X_{11}, \dots, X_{nk_n}; Z)_h \end{array}$$

for objects $Z \in \mathcal{M}_{s_2}$, $Y_i \in \mathcal{M}_{s_1}$ and $X_{ij} \in \mathcal{M}_{s_0}$. We can interpret these maps as a map of (coloured) symmetric sequences from the composition product of two bimodules to the third bimodule. Associativity of composition in the ∞-category $\mathcal{M}^{\otimes}$ implies that this map factors through a relative composition product over the 'middle' ∞-operad $\mathcal{M}^{\otimes}_{s_1}$.

The aim of this section is to construct families of ∞-operads that capture the Goodwillie derivatives of a diagram of ∞-categories, together with all the bimodule structures that those derivatives possess. More precisely:

Goal 6.6. Let S be a simplicial set, and let

$$p : \mathcal{X} \to S^{op}$$

be a cartesian fibration that encodes a diagram of pointed compactly-generated ∞-categories, $\mathcal{X}_s$ for each vertex $s \in S$, and reduced finitary functors, $\mathcal{X}_f : \mathcal{X}_s \to \mathcal{X}_{s'}$ for each edge $f : s \to s'$ in S. We will construct from p a corresponding stable family of ∞-operads

$$p_0 : \mathbb{D}^{\otimes}_{\mathcal{X}} \to S \times \mathcal{S}\mathrm{urj}_*$$

with the following properties:

- for each vertex $s \in S$, the fibre
$$(p_0)_s : (\mathbb{D}^{\otimes}_{\mathcal{X}})_s \to \mathcal{S}\mathrm{urj}_*$$
is isomorphic to the stable ∞-operad $\mathbb{I}^{\otimes}_{\mathcal{X}_s}$ associated to the pointed compactly-generated ∞-category $\mathcal{X}_s$, i.e. this fibre encodes the derivatives of the identity functor on $\mathcal{X}_s$;
- for each edge $f : s \to s'$ in S, the bimodule
$$(p_0)_f : (\mathbb{D}^{\otimes}_{\mathcal{X}})_f \to \Delta^1 \times \mathcal{S}\mathrm{urj}_*$$
encodes the derivatives of the reduced finitary functor $\mathcal{X}_f : \mathcal{X}_s \to \mathcal{X}_{s'}$.

In particular, when the diagram consists of a single functor $F : \mathcal{C} \to \mathcal{D}$, the stable family will be a single $(\mathbb{I}^{\otimes}_{\mathcal{D}}, \mathbb{I}^{\otimes}_{\mathcal{C}})$-bimodule $\mathbb{D}^{\otimes}_F$ that consists of the derivatives of F.

When the diagram consists of a pair of functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$, and their composite $GF : \mathcal{C} \to \mathcal{E}$, we obtain the stable family of ∞-operads that encodes the three bimodules given by the derivatives of these three functors, together with a map of $(\mathbb{I}^{\otimes}_{\mathcal{E}}, \mathbb{I}^{\otimes}_{\mathcal{C}})$-bimodules of the form

$$\mathbb{D}^{\otimes}_G \circ_{\mathbb{I}^{\otimes}_{\mathcal{D}}} \mathbb{D}^{\otimes}_F \to \mathbb{D}^{\otimes}_{GF}.$$

In 6.26 we conjecture a Chain Rule which claims that this map of bimodules is an equivalence.

Remark 6.7. Here is the idea behind our construction of $\mathbb{D}^{\otimes}_{\mathcal{X}}$ in the case where $p : \mathcal{X} \to (\Delta^1)^{op}$ is a cartesian fibration that represents a single reduced finitary functor $F : \mathcal{C} \to \mathcal{D}$.

The functor F determines, by precomposition, a functor

$$F^* : \mathcal{F}_{\mathcal{D}} \to \mathcal{F}_{\mathcal{C}}; \quad G \mapsto GF$$

which is symmetric monoidal with respect to the pointwise smash product.

The functor F^* in turn induces a *lax* symmetric monoidal functor (i.e. a map of ∞-operads) between the Day convolution monoidal structures:

$$F_* : \mathrm{Fun}(\mathcal{F}_{\mathcal{C}}, \mathcal{S}p)^{\otimes} \to \mathrm{Fun}(\mathcal{F}_{\mathcal{D}}, \mathcal{S}p)^{\otimes}; \quad \mathsf{A}(-) \mapsto \mathsf{A}(-F).$$

The functor F_* has a left adjoint $F^!$ given by left Kan extension along F^*, and this left adjoint is a symmetric monoidal functor

$$F^! : \mathrm{Fun}(\mathcal{F}_{\mathcal{D}}, \mathcal{S}p)^{\otimes} \to \mathrm{Fun}(\mathcal{F}_{\mathcal{C}}, \mathcal{S}p)^{\otimes}.$$

Taking opposites, we get a symmetric monoidal functor

$$(F^!)^{op} : \mathrm{Fun}(\mathcal{F}_{\mathcal{D}}, \mathcal{S}p)^{op,\otimes} \to \mathrm{Fun}(\mathcal{F}_{\mathcal{C}}, \mathcal{S}p)^{op,\otimes}.$$

Pulling back the left action of $\mathrm{Fun}(\mathcal{F}_{\mathcal{C}}, \mathcal{S}p)^{op,\otimes}$ on itself along the map $(F^!)^{op}$, we get a $(\mathrm{Fun}(\mathcal{F}_{\mathcal{D}}, \mathcal{S}p)^{op,\otimes}, \mathrm{Fun}(\mathcal{F}_{\mathcal{C}}, \mathcal{S}p)^{op,\otimes})$-bimodule $\mathcal{M}^{\otimes}$ whose multi-morphism spectra are given by

$$\mathrm{Map}_{\mathcal{M}^{\otimes}}(A_1, \dots, A_n; B) = \mathrm{Nat}_{\mathcal{F}_{\mathcal{C}}}(F^! B, A_1 \otimes \cdots \otimes A_n).$$

Restricting to the ∞-operads $\mathbb{I}^{\otimes}_{\mathcal{D}}$ and $\mathbb{I}^{\otimes}_{\mathcal{C}}$, we get a bimodule $\mathbb{D}^{\otimes}_{\mathcal{X}}$ whose multi-morphism spectra are

$$\mathrm{Map}_{\mathbb{D}^{\otimes}_{\mathcal{X}}}(X_1, \dots, X_n; Y) \simeq \mathrm{Nat}_{\mathcal{F}_{\mathcal{C}}}(F^! \partial_1(-)(Y); \partial_1(-)(X_1) \otimes \cdots \otimes \partial_1(-)(X_n))$$

which, via the adjunction property of the left Kan extension $F^!$ and Theorems 2.4 and 3.1, is equivalent to the derivative

$$\partial_n F(X_1, \dots, X_n; Y).$$

Now suppose we are given an S-indexed diagram $\mathcal{X}$ of pointed compactly-generated ∞-categories and reduced finitary functors. Then the above procedure yields an S^{op}-indexed diagram of stable symmetric monoidal ∞-categories $\mathrm{Fun}(\mathcal{F}_{\mathcal{X}_s}, \mathcal{S}p)^{op,\otimes}$ and exact symmetric monoidal functors $(\mathcal{X}^!_f)^{op}$. Such a diagram can be expressed as an S-family of stable ∞-operads

$$(*) \qquad \mathrm{Fun}(\mathcal{F}_{\mathcal{X}}, \mathcal{S}p)^{op,\otimes} \to S \times \mathcal{S}\mathrm{urj}_*$$

whose fibre over $s \in S$ is the symmetric monoidal ∞-category

$$\mathrm{Fun}(\mathcal{F}_{\mathcal{X}_s}, \mathcal{S}p)^{op,\otimes} \to \mathcal{S}\mathrm{urj}_*.$$

(Note the reversal here: an S^{op}-indexed diagram of symmetric monoidal functors is encoded as an S-family of ∞-operads.) Finally, the desired S-family of ∞-operads

$$\mathbb{D}^{\otimes}_{\mathcal{X}} \to S \times \mathcal{S}\mathrm{urj}_*$$

is the restriction of (*) to the full subcategory whose fibre over s is the ∞-operad $\mathbb{I}^{\otimes}_{\mathcal{X}_s} \subseteq \mathrm{Fun}(\mathcal{F}_{\mathcal{X}_s}, \mathcal{S}p)^{op,\otimes}$.

Let us now carry out the precise construction envisaged in Remark 6.7. To some extent this process amounts to a fibrewise (over S) version of the construction of the ∞-operad $\mathbb{I}^{\otimes}_{\mathcal{C}}$ in Section 4, paying attention to how the functoriality is represented in families of symmetric monoidal ∞-categories and symmetric monoidal functors. For this purpose, we will use fibrewise versions of the three main tools involved in that construction, and two of these depend on the following fibrewise mapping space construction.

Definition 6.8. Let $X \to S$ and $Y \to S$ be maps of simplicial sets. We define $\mathrm{Fun}_S(X, Y)$ to be the simplicial set over S for which an n-simplex consists of:

- an n-simplex $\Delta^n \to S$;
- a map $\Delta^n \times_S X \to Y$ over S;

with simplicial structure given by precomposition with simplex maps $\Delta^m \to \Delta^n$ in a standard way. The construction $\mathrm{Fun}_S(-,-)$ is the internal mapping object in the cartesian closed category of simplicial sets over S.

Here is our fibrewise version of Construction 4.11.

Construction 6.9. Let $p : \mathcal{X} \to S^{op}$ be a cartesian fibration. Define a pullback of simplicial sets

$$\begin{array}{ccc} \mathrm{Fun}_{S^{op}}(\mathcal{X}, \mathcal{S}p)^{\wedge} & \longrightarrow & \mathrm{Fun}_{S^{op}}(\mathcal{X}, S^{op} \times \mathcal{S}p^{\wedge}) \\ {\scriptstyle p'}\downarrow & & \downarrow \\ S^{op} \times \mathcal{S}\mathrm{urj}_* & \longrightarrow & \mathrm{Fun}_{S^{op}}(\mathcal{X}, S^{op} \times \mathcal{S}\mathrm{urj}_*) \end{array} \tag{6.10}$$

where the right-hand map is induced by the cocartesian fibration $\mathcal{S}p^{\wedge} \to \mathcal{S}\mathrm{urj}_*$, and the bottom map sends an n-simplex $\Delta^n \to S^{op} \times \mathcal{S}\mathrm{urj}_*$ to the composite

$$\Delta^n \times_{S^{op}} \mathcal{X} \to \Delta^n \to S^{op} \times \mathcal{S}\mathrm{urj}_*.$$

Proposition 6.11. *Let $p : \mathcal{X} \to S^{op}$ be a cartesian fibration. Then the map*

$$p' : \mathrm{Fun}_{S^{op}}(\mathcal{X}, \mathcal{S}p)^{\wedge} \to S^{op} \times \mathcal{S}\mathrm{urj}_*$$

of Construction 6.9 is a stable S^{op}-family of ∞-operads with the following properties:

- *the map p' is a cocartesian fibration;*
- *the fibre of p' over $s \in S$ is the symmetric monoidal ∞-category $\mathrm{Fun}(\mathcal{X}_s, \mathcal{S}p)^{\wedge}$ of Construction 4.11;*
- *the multi-morphism spectra for p', over $f : s \to s'$ in S, are given by*

  $$\mathrm{Map}_{\mathrm{Fun}_{S^{op}}(\mathcal{X},\mathcal{S}p)^{\wedge}}(F_1, \dots, F_n; G)_f \simeq \mathrm{Nat}_{\mathcal{X}_s}(\mathcal{X}_f^*(F_1) \wedge \dots \wedge \mathcal{X}_f^*(F_n), G)$$

  where $\mathcal{X}_f^*$ denotes precomposition with the functor $\mathcal{X}_f : \mathcal{X}_s \to \mathcal{X}_{s'}$ in the diagram of ∞-categories classified by the cartesian fibration p.

Proof. Firstly, it follows from [17, 3.2.2.12] that the right-hand vertical map in (6.10) is a cocartesian fibration, hence the pullback p' is too.

We now check conditions (1)-(3) of Definition 6.1. Comparing with Construction 4.11, the fibres of p' are indeed the symmetric monoidal ∞-categories $\mathrm{Fun}(\mathcal{X}_s, \mathcal{S}p)$, and so are stable ∞-operads, satisfying (1).

Since p' is a cocartesian fibration, the p'_s-cocartesian lifts of inert morphisms in $\mathcal{S}\mathrm{urj}_*$ are also p'-cocartesian, which gives (2). We also know that p' classifies an $S^{op} \times \mathcal{S}\mathrm{urj}_*$-indexed diagram of ∞-categories of the form

$$(s, \langle n \rangle) \mapsto \mathrm{Fun}(\mathcal{X}_s, \mathcal{S}p^{\wedge}_{\langle n \rangle})$$

where the arrows in this diagram are those induced by pulling back along $\mathcal{X}_f : \mathcal{X}_{s'} \to \mathcal{X}_s$, for edges $f : s \to s'$ in S, and by the functors

$$\bar{\alpha} : \mathcal{S}p^{\wedge}_{\langle n \rangle} \to \mathcal{S}p^{\wedge}_{\langle n' \rangle}$$

associated to a morphism $\alpha : \langle n \rangle \to \langle n' \rangle$ by the cocartesian fibration $\mathcal{S}p^{\wedge} \to \mathcal{S}\mathrm{urj}_*$.

The mapping space in the ∞-category $\mathrm{Fun}_{S^{op}}(\mathcal{X},\mathcal{S}p)^\wedge$, over some morphism

$$(f,\alpha):(s,\langle n\rangle)\to(s',\langle n'\rangle)$$

in $S^{op}\times\mathcal{S}\mathrm{urj}_*$, are then given by

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{Fun}_{S^{op}}(\mathcal{X},\mathcal{S}p)^\wedge}((F_1,\dots,F_n),(G_1,\dots,G_{n'}))_{(f,\alpha)}\\ &\qquad\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{X}_{s'},\mathcal{S}p^\wedge_{\langle n'\rangle})}(\bar{\alpha}(F_1,\dots,F_n)\mathcal{X}_f,(G_1,\dots,G_{n'})).\end{aligned}$$

Since $\mathrm{Fun}(\mathcal{X}_{s'},\mathcal{S}p^\wedge_{\langle n'\rangle})\simeq\mathrm{Fun}(\mathcal{X}_{s'},\mathcal{S}p)^{n'}$, this formula yields condition (3) of Definition 6.1 and gives us the multi-morphism spaces

$$\mathrm{Hom}_{\mathrm{Fun}_{S^{op}}(\mathcal{X},\mathcal{S}p)^\wedge}(F_1,\dots,F_n;G)_f\simeq\mathrm{Hom}_{\mathrm{Fun}(\mathcal{X}_{s'},\mathcal{S}p)}(\mathcal{X}_f^*(F_1)\wedge\dots\wedge\mathcal{X}_f^*(F_n);G).$$

Since $\mathcal{X}_f^*$ is an exact functor and $\mathrm{Fun}(\mathcal{X}_{s'},\mathcal{S}p)$ is a stable ∞-category, we get the second part of Definition 6.2, so p' is a stable S^{op}-family of ∞-operads and hence also the desired formulas for the multi-morphism spectra. ☐

Let us introduce the following terminology for the type of ∞-operad family described in Proposition 6.11.

Definition 6.12. Let S be a simplicial set. A *cocartesian S-family of symmetric monoidal ∞-categories* is an S-family of ∞-operads

$$p:\mathcal{C}^\otimes\to S\times\mathcal{S}\mathrm{urj}_*$$

for which the structure map p is a cocartesian fibration.

Remark 6.13. A cocartesian S-family of symmetric monoidal ∞-categories p corresponds to an S-indexed diagram of symmetric monoidal ∞-categories $\mathcal{C}_s^\otimes\to\mathcal{S}\mathrm{urj}_*$ and symmetric monoidal functors $\mathcal{C}_f:\mathcal{C}_s\to\mathcal{C}_{s'}$ for each edge $f:s\to s'$ in S. Such a family is stable if and only if each $\mathcal{C}_s$ is stable and each $\mathcal{C}_f$ is exact. In this case the multi-morphism spectra are given by

$$\mathrm{Map}_{\mathcal{C}^\otimes}(X_1,\dots,X_n;Y)_f\simeq\mathrm{Map}_{\mathcal{C}_{s'}}(\mathcal{C}_f(X_1)\otimes\cdots\otimes\mathcal{C}_f(X_n),Y).$$

Definition 6.14. Now suppose that the cartesian fibration $p:\mathcal{X}\to S^{op}$ encodes a diagram of pointed compactly-generated ∞-categories and reduced finitary functors. Let

$$p'':\mathcal{F}^\wedge_{\mathcal{X}}\to S^{op}\times\mathcal{S}\mathrm{urj}_*$$

be the restriction of the map p' of Construction 6.9 to the full subcategory $\mathcal{F}^\wedge_{\mathcal{X}}\subseteq\mathrm{Fun}_{S^{op}}(\mathcal{X},\mathcal{S}p)^\wedge$ generated by the reduced finitary functors $\mathcal{X}_s\to\mathcal{S}p$ for $s\in S$. Then p'' is also a stable cocartesian S^{op}-family of symmetric monoidal ∞-categories because each fibre is the stable symmetric monoidal ∞-category $\mathcal{F}_{\mathcal{X}_s}$, and the functors $\mathcal{X}_f^*$ restrict to exact symmetric monoidal functors $\mathcal{F}_{\mathcal{X}_{s'}}\to\mathcal{F}_{\mathcal{X}_s}$.

Recall that since each $\mathcal{F}_{\mathcal{X}_s}$ is not small, we have to restrict further to compact objects before applying the Day convolution. These objects are not necessarily preserved by the pullback functors $\mathcal{X}_f^*$ so we have to enlarge the subcategories of compact objects to include all those functors obtained by pulling back a compact object.

Definition 6.15. Let $p : \mathcal{X} \to S^{op}$ and $p'' : \mathcal{F}_{\mathcal{X}}^{\wedge} \to S^{op} \times \mathcal{S}\mathrm{urj}_*$ be as in Definition 6.14, and let

$$p_1 : (\hat{\mathcal{F}}_{\mathcal{X}}^{\omega})^{\wedge} \to S^{op} \times \mathcal{S}\mathrm{urj}_*$$

be the restriction of p'' to the symmetric monoidal subcategories generated by those reduced finitary functors $\mathcal{X}_s \to \mathcal{S}p$ that are of the form $\mathcal{X}_f^*(G)$ for some compact object $G \in \mathcal{F}_{\mathcal{X}_{s'}}^{\omega}$ and some $f : s \to s'$ in S.

Proposition 6.16. *The map $p_1 : (\hat{\mathcal{F}}_{\mathcal{X}}^{\omega})^{\wedge} \to S^{op} \times \mathcal{S}\mathrm{urj}_*$ is a stable cocartesian S^{op}-family of symmetric monoidal ∞-categories for which each fibre $(\hat{\mathcal{F}}_{\mathcal{X}}^{\omega})_s$ is essentially small and generates $\mathcal{F}_{\mathcal{X}_s}$ under filtered colimits.*

We now wish to apply to p_1 a fibrewise version of the Day convolution symmetric monoidal ∞-category of Construction 4.13. We will delay the details of this version to Appendix B; the following proposition captures what we need from it.

Proposition 6.17. *Let $p_1 : \mathcal{C}^{\otimes} \to S^{op} \times \mathcal{S}\mathrm{urj}_*$ be a stable cocartesian S^{op}-family of symmetric monoidal ∞-categories such that each fibre $\mathcal{C}_s$ is essentially small. Let $q : \mathcal{D}^{\otimes} \to \mathcal{S}\mathrm{urj}_*$ be a stable symmetric monoidal ∞-category for which the monoidal product commutes with colimits in each variable. Then there is a stable cocartesian S^{op}-family of symmetric monoidal ∞-categories*

$$p_2 : \mathrm{Fun}_{S^{op}}(\mathcal{C}, \mathcal{D})^{\otimes} \to S^{op} \times \mathcal{S}\mathrm{urj}_*$$

such that

- *the fibre of p_2 over $s \in S$ is the Day convolution symmetric monoidal ∞-category*
$$(p_2)_s : \mathrm{Fun}(\mathcal{C}_s, \mathcal{D})^{\otimes} \to \mathcal{S}\mathrm{urj}_*$$
of Construction 4.13;
- *the multi-morphism spectra over $f : s \to s'$ in S are given by*
$$\begin{aligned}\mathrm{Map}_{\mathrm{Fun}_{S^{op}}(\mathcal{C},\mathcal{D})^{\otimes}}(A_1, \dots, A_n; B)_f &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_{s'},\mathcal{D})}(\mathcal{C}_f^!(A_1) \otimes \cdots \otimes \mathcal{C}_f^!(A_n), B) \\ &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_{s'},\mathcal{D})}(\mathcal{C}_f^!(A_1 \otimes \cdots \otimes A_n), B) \\ &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_s,\mathcal{D})}(A_1 \otimes \cdots \otimes A_n, B\mathcal{C}_f)\end{aligned}$$
for $A_1, \dots, A_n : \mathcal{C}_s \to \mathcal{D}$ and $B : \mathcal{C}_{s'} \to \mathcal{D}$. Here $\mathcal{C}_f^!$ denotes left Kan extension along the symmetric monoidal functor $\mathcal{C}_s \to \mathcal{C}_{s'}$ associated to f by the cocartesian family p_1.

Definition 6.18. Let $p : \mathcal{X} \to S^{op}$ be a cartesian fibration as in Definition 6.14, and let $p_1 : \hat{\mathcal{F}}_{\mathcal{X}}^{\omega} \to S^{op} \times \mathcal{S}\mathrm{urj}_*$ be as in Definition 6.15. Applying Proposition 6.17 with this p_1 and with $q : \mathcal{S}p^{\wedge} \to \mathcal{S}\mathrm{urj}_*$ the usual (non-unital) symmetric monoidal ∞-category of spectra under the smash product, we obtain a stable cocartesian S^{op}-family of symmetric monoidal ∞-categories

$$p_2 : \mathrm{Fun}_{S^{op}}(\hat{\mathcal{F}}_{\mathcal{X}}^{\omega}, \mathcal{S}p)^{\otimes} \to S^{op} \times \mathcal{S}\mathrm{urj}_*.$$

The final step is to apply a fibrewise version of Construction 4.17. The details of this construction are in Appendix C; here is what we need from it.

Proposition 6.19. *Let $p_2 : \mathcal{M}^{\otimes} \to S^{op} \times \mathcal{S}\mathrm{urj}_*$ be a stable cocartesian S^{op}-family of symmetric monoidal ∞-categories corresponding to a diagram of exact symmetric monoidal functors $\mathcal{M}^{\otimes}_f : \mathcal{M}^{\otimes}_{s'} \to \mathcal{M}^{\otimes}_s$ between stable symmetric monoidal ∞-categories. Then there is a stable S-family of symmetric monoidal ∞-categories*

$$p_3 : \mathcal{M}^{op,\otimes} \to S \times \mathcal{S}\mathrm{urj}_*$$

such that:

- *for each $s \in S$, the fibre $\mathcal{M}^{op,\otimes}_s \to \mathcal{S}\mathrm{urj}_*$ is isomorphic to the opposite symmetric monoidal ∞-category of $\mathcal{M}^{\otimes}_s$, as in Construction 4.17;*
- *for each $f : s \to s'$ in S, we have multi-morphism spectra*

$$\mathrm{Map}_{\mathcal{M}^{op,\otimes}}(Y_1, \dots, Y_n; X) \simeq \mathrm{Map}_{\mathcal{M}_s}(\mathcal{M}_f(X), Y_1 \otimes \dots \otimes Y_n).$$

Definition 6.20. Let $p : \mathcal{X} \to S^{op}$ be a cartesian fibration as in Definition 6.14, and let $p_2 : \mathrm{Fun}_{S^{op}}(\hat{\mathfrak{F}}^{\omega}_{\mathcal{X}}, \mathcal{S}p)^{\otimes} \to S^{op} \times \mathcal{S}\mathrm{urj}_*$ be as in Definition 6.18. Applying Proposition 6.19, we obtain a stable S-family of symmetric monoidal ∞-categories

$$p_3 : \mathrm{Fun}_{S^{op}}(\hat{\mathfrak{F}}^{\omega}_{\mathcal{X}}, \mathcal{S}p)^{op,\otimes} \to S \times \mathcal{S}\mathrm{urj}_*$$

whose fibre over $s \in S$ is the symmetric monoidal ∞-category

$$\mathrm{Fun}(\hat{\mathfrak{F}}^{\omega}_{\mathcal{X}_s}, \mathcal{S}p)^{op,\otimes} \to \mathcal{S}\mathrm{urj}_*$$

and with multi-morphism spectra

$$\mathrm{Map}_{\hat{\mathfrak{F}}^{\omega}_{\mathcal{X}_s}}(\mathcal{X}^!_f(A), B_1 \otimes \dots \otimes B_n)$$

for $A : \hat{\mathfrak{F}}^{\omega}_{\mathcal{X}_{s'}} \to \mathcal{S}p$ and $B_1, \dots, B_n : \hat{\mathfrak{F}}^{\omega}_{\mathcal{X}_s} \to \mathcal{S}p$, where $\mathcal{X}^!_f$ is the left Kan extension along the functor $\mathcal{X}^*_f$.

Definition 6.21. With p and p_3 as in Definition 6.20, let $\mathbb{D}^{\otimes}_{\mathcal{X}}$ be the full subcategory of $\mathrm{Fun}_{S^{op}}(\hat{\mathfrak{F}}^{\omega}_{\mathcal{X}}, \mathcal{S}p)^{op,\otimes}$ whose objects over $s \in S$ are those of the ∞-operad $\mathbb{I}^{\otimes}_{\mathcal{X}_s}$. Let

$$p_0 : \mathbb{D}^{\otimes}_{\mathcal{X}} \to S \times \mathcal{S}\mathrm{urj}_*$$

be the restriction of p_3 to this subcategory. Recall that the objects of $\mathbb{I}^{\otimes}_{\mathcal{X}_s}$ are the functors

$$\partial_1(-)(X) : \mathcal{F}_{\mathcal{X}_s} \to \mathcal{S}p$$

for $X \in \mathcal{S}p(\mathcal{X}_s)$, and that we usually denote this object just by X.

Theorem 6.22. *Let $p : \mathcal{X} \to S^{op}$ be a cartesian fibration that encodes a diagram of pointed compactly-generated ∞-categories and reduced finitary functors. Then the map*

$$p_0 : \mathbb{D}^{\otimes}_{\mathcal{X}} \to S \times \mathcal{S}\mathrm{urj}_*$$

is a stable S-family of ∞-operads whose fibre over s is the stable ∞-operad $\mathbb{I}^{\otimes}_{\mathcal{X}_s}$, and for which the bimodule corresponding to a morphism $f : s \to s'$ in S has multi-morphism spectra given by

$$\mathrm{Map}_{\mathbb{D}^{\otimes}_{\mathcal{X}}}(X_1, \dots, X_n; Y)_f \simeq \partial_n(\mathcal{X}_f)(X_1, \dots, X_n; Y)$$

for $X_1,\dots,X_n \in \mathcal{S}p(\mathcal{X}_s)$ *and* $Y \in \mathcal{S}p(\mathcal{X}_{s'})$*. Here* $\mathcal{X}_f : \mathcal{X}_s \to \mathcal{X}_{s'}$ *is the functor classified by the cartesian fibration* p *over* f*.*

Proof. The fact that p_0 is a stable S-family of ∞-operads follows from the fact that p_3 is, and that each $\mathbb{I}^{\otimes}_{\mathcal{X}_s}$ is a stable ∞-operad. It remains to identify the multi-morphism spectra. These are a priori given by

$$\mathrm{Nat}_{\hat{\mathcal{F}}^{\omega}_{\mathcal{X}_s}}(\partial_1(\mathcal{X}_f^*(-))(Y), \partial_1(-)(X_1) \otimes \dots \otimes \partial_1(-)(X_n))$$

which can be written as

$$\mathrm{Nat}_{\hat{\mathcal{F}}^{\omega}_{\mathcal{X}_s}}(\partial_1(-\mathcal{X}_f)(Y), \partial_1(-)(X_1) \otimes \dots \otimes \partial_1(-)(X_n)).$$

The argument of Proposition 4.20 tells us that both the Day convolution and the natural transformation object can be calculated over $\mathcal{F}_{\mathcal{X}_s}$ instead of $\hat{\mathcal{F}}^{\omega}_{\mathcal{X}_s}$. It then follows from Theorems 2.4 and 3.1 that the multi-morphism spectra are given by

$$\partial_n(\mathcal{X}_f)(X_1,\dots,X_n;Y)$$

as claimed. □

Definition 6.23. Let $F : \mathcal{C} \to \mathcal{D}$ be a reduced finitary functor between pointed compactly-generated ∞-categories. We write

$$\mathbb{D}^{\otimes}_F \to \Delta^1 \times \mathcal{S}\mathrm{urj}_*$$

for the $(\mathbb{I}^{\otimes}_{\mathcal{D}}, \mathbb{I}^{\otimes}_{\mathcal{C}})$-bimodule given by applying Definition 6.21 in the case that p is the cartesian fibration $\mathcal{X} \to (\Delta^1)^{op}$ corresponding to the functor F. The multi-morphism spectra of the bimodule $\mathbb{D}^{\otimes}_F$ are then simply the derivatives of F.

Proposition 6.24. *The stable* S*-family of* ∞*-operads of Definition 6.21 is corepresentable (i.e. a locally cocartesian fibration) via the functors*

$$\boldsymbol{\Delta}_n(\mathcal{X}_f) : \mathcal{S}p(\mathcal{X}_s)^n \to \mathcal{S}p(\mathcal{X}_{s'})$$

of Section 1.

Proof. From Theorem 6.22 and Definition 1.1 we have

$$\begin{aligned}\mathrm{Map}_{\mathbb{D}^{\otimes}_{\mathcal{X}}}(X_1,\dots,X_n;Y)_f &\simeq \partial_n(\mathcal{X}_f)(X_1,\dots,X_n;Y)\\ &\simeq \mathrm{Map}_{\mathcal{S}p(\mathcal{X}_{s'})}(Y, \boldsymbol{\Delta}_n(\mathcal{X}_f)(X_1,\dots,X_n))\end{aligned}$$

which implies the claim. □

Remark 6.25. Proposition 6.24 can be interpreted as providing for the existence of maps of the form

$$\boldsymbol{\Delta}_k G(\boldsymbol{\Delta}_{n_1}F,\dots,\boldsymbol{\Delta}_{n_k}F) \to \boldsymbol{\Delta}_{n_1+\dots+n_k}(GF)$$

that make the construction $\boldsymbol{\Delta}_*$ lax monoidal (up to higher coherent homotopies).

We conclude with a conjectured chain rule that generalizes that of Arone and the author for the ∞-categories of based spaces and spectra [1].

Conjecture 6.26. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be reduced finitary functors between pointed compactly-generated ∞-categories. Let $\mathcal{X} \to (\Delta^2)^{op}$ be the cartesian fibration that encodes the diagram consisting of the functors F, G and GF, and let*

$$\mathbb{D}^{\otimes}_{\mathcal{X}} \to \Delta^2 \times \mathcal{S}\mathrm{urj}_*$$

be the associated Δ^2-family of ∞-operads of Theorem 6.22. Then the corresponding map, described in Remark 6.5, of $(\mathbb{I}^{\otimes}_{\mathcal{E}}, \mathbb{I}^{\otimes}_{\mathcal{C}})$-bimodules,

$$\mathbb{D}^{\otimes}_{G} \circ_{\mathbb{I}^{\otimes}_{\mathcal{D}}} \mathbb{D}^{\otimes}_{F} \to \mathbb{D}^{\otimes}_{GF}$$

is an equivalence.

One of the challenges in proving 6.26 is identifying the underlying property of a stable Δ^2-family of ∞-operads which implies that this associated map of bimodules is an equivalence. This property should be some version of Lurie's 'flatness' condition [18, B.3]. Notice that [18, 3.1.4.2] describes a consequence of that assumption in the unstable case. This result suggests that we describe the chain rule in terms of induced functors between ∞-categories of algebras. In particular, it seems that any stable S-family of ∞-operads should determine a locally cartesian fibration that classifies those induced functors. The Chain Rule would then follow from showing that this locally cartesian fibration is in fact cartesian in the case at hand.

We note further, however, that the proof of the Chain Rule in [1] depends considerably on Koszul duality between the operad $\partial_* I_{\mathcal{C}}$ and the cooperad formed by the derivatives of the functor $\Sigma^\infty \Omega^\infty$. Since that duality does not play an explicit role in the current constructions, we might expect to need a new argument here.

Finally, we should note that Lurie proves a version of Conjecture 6.26 in [18, 6.3.2] for *coderivatives* instead of derivatives. We expect the operad theory developed here to be Koszul dual, in a suitable sense, to Lurie's, in which case we might expect to deduce 6.26 directly from Lurie's results.

Appendix A. The chain rule for spectrum-valued functors

In the proof of Theorem 3.1 we needed a chain rule for composites of functors $G : \mathcal{C} \to \mathcal{S}p$ and $F : \mathcal{S}p \to \mathcal{S}p$. The purpose of this section is to state and prove the needed result, which is a mild generalization of [7, 1.15].

Theorem A.1. *Let $\mathcal{C}$ be a pointed compactly-generated ∞-category and let $G : \mathcal{C} \to \mathcal{S}p$ and $F : \mathcal{S}p \to \mathcal{S}p$ be reduced functors. Assume that F preserves filtered homotopy colimits. Then for $X_1, \dots, X_n \in \mathcal{S}p(\mathcal{C})$ we have*

$$\begin{array}{c} \partial_n(FG)(X_1,\dots,X_n) \\ \sim\downarrow \\ \displaystyle\prod_{\mu \in \mathsf{P}(n)} \partial_k(F) \wedge \partial_{n_1}(G)(\{X_i\}_{i\in\mu_1}) \wedge \dots \wedge \partial_{n_k}(G)(\{X_i\}_{i\in\mu_k}) \end{array}$$

where the product is over the set $\mathsf{P}(n)$ *of* unordered *partitions* μ *of* $\{1,\dots,n\}$ *into* k *pieces* $\mu_1,\dots,\mu_k$*, with* $n_j = |\mu_j|$*.*

Proof. We follow the approach of [7] very closely. Indeed, many of the results proved there carry over to this more general situation with no change. Specifically, we can construct, as in [7, 2.5], a map

$$\Delta : FG \to \prod_{\lambda} [P_{k_1},\dots,P_{k_r}]\,\mathrm{cr}_r(F)(P_{l_1}G,\dots,P_{l_r}G)$$

where λ varies over expressions of the form

$$n = k_1 l_1 + \dots + k_r l_r.$$

with k_i and l_i positive integers such that $l_1 < \dots < l_r$. We can also prove, as in [7, 4.2], that Δ induces an equivalence on D_n, and hence on n^{th} derivatives. Moreover, we can show, as in the proof of [7, 2.6], that the n^{th} derivative of the functor

$$[P_{k_1},\dots,P_{k_r}]\,\mathrm{cr}_r(F)(P_{l_1}G,\dots,P_{l_r}G)$$

is equivalent to the n^{th} derivative of the n-homogeneous functor

$$(*) \qquad (\partial_k F \wedge (D_{l_1}G)^{\wedge k_1} \wedge \dots \wedge (D_{l_r}G)^{\wedge k_r})_{h\Sigma_{k_1}\times\dots\times\Sigma_{k_r}}$$

where $k = k_1 + \dots + k_r$. It now remains to calculate this n^{th} derivative at an n-tuple $(X_1,\dots,X_n)$ in $\mathcal{S}p(\mathcal{C})$.

Since all the functors involved here are homogeneous, and thus factor via $\Sigma^\infty_{\mathcal{C}} : \mathcal{C} \to \mathcal{S}p(\mathcal{C})$, we can assume without loss of generality that $\mathcal{C}$ is stable. Using the equivalence

$$D_l G(X) \simeq \partial_l G(X,\dots,X)_{h\Sigma_l}$$

we can write the functor (*) as mapping X to

$$(\partial_k F \wedge \partial_{l_1} G(X,\dots,X)^{\wedge k_1} \wedge \dots \wedge \partial_{l_r} G(X,\dots,X)^{\wedge k_r})_{hH(\lambda)}$$

where $H(\lambda)$ denotes the subgroup $(\Sigma_{l_1} \wr \Sigma_{k_1}) \times \dots \times (\Sigma_{l_r} \wr \Sigma_{k_r})$ of Σ_n formed from wreath products. It's convenient to rewrite this as

$$(\partial_k F \wedge \partial_{n_1} G(X,\dots,X) \wedge \dots \wedge \partial_{n_k} G(X,\dots,X))_{hH(\lambda)}$$

where $n_1,\dots,n_k$ are the numbers $l_1,\dots,l_r$ with l_i repeated k_i times.

Now when $E : \mathcal{C}^n \to \mathcal{S}p$ is a multilinear functor, the n^{th} derivative of the functor $X \mapsto E(X,\dots,X)$ at $(X_1,\dots,X_n)$ can be written as

$$\prod_{\sigma\in\Sigma_n} E(X_{\sigma(1)},\dots,X_{\sigma(n)})$$

It follows from all of this that $\partial_n(FG)(X_1,\dots,X_n)$ can be expressed as

$$\prod_{\lambda} \left(\prod_{\sigma\in\Sigma_n} \partial_k F \wedge \partial_{n_1} G(X_{\sigma(1)},\dots,X_{\sigma(n_1)}) \wedge \dots \wedge \partial_{n_k} G(X_{\sigma(n-n_k+1)},\dots,X_{\sigma(n)}) \right)_{hH(\lambda)}.$$

It remains to identify this with the formula stated in the Theorem. We do this by showing that a choice of expression λ, together with a coset $[\sigma]$ of $H(\lambda)$ in Σ_n, uniquely corresponds to an unordered partition of $\{1,\dots,n\}$.

In one direction, we map the pair $(\lambda,[\sigma])$ to the partition whose pieces are the sets $(\sigma(1),\dots,\sigma(n_1)),\dots,(\sigma(n-n_k+1),\dots,\sigma(n))$. On the other hand, given an unordered partition μ, let k_j be the number of pieces of size l_j (determining λ). If we put the pieces of μ in ascending size order, and concatenate them, we get an element $\sigma \in \Sigma_n$ which determines a coset of $H(\lambda)$. This is well-defined because changing the order of elements within each piece, or the order of pieces of the same size, only changes σ by an element of $H(\lambda)$. It is a simple check that these two constructions are inverse, setting up the desired correspondence. Via this bijection, the expression given above for $\partial_n(FG)(X_1,\dots,X_n)$ corresponds with the desired formula. □

Appendix B. Fibrewise Day convolution

The goal of this section is to prove Proposition 6.17 which provides the fibrewise version of Day convolution needed in the construction of the family of ∞-operads $\mathbb{D}^{\otimes}_{\mathcal{X}}$ that captures the derivatives of the functors in a diagram $\mathcal{X}$ of pointed compactly-generated ∞-categories.

The underlying idea of this construction is that the Day convolution $\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}$ for two symmetric monoidal ∞-categories is functorial in its first variable with respect to symmetric monoidal functors. This fact was shown by Nikolaus in [19, 3.8]. When $\mathcal{D}^{\otimes}$ is presentably symmetric monoidal, the resulting pullback functor is only lax symmetric monoidal, but has a left adjoint, an operadic left Kan extension, which is (strong) symmetric monoidal. Proposition 6.17 captures this functoriality.

So now let $p_1 : \mathcal{C}^{\otimes} \to S^{op} \times \mathcal{S}\mathrm{urj}_*$ be a stable cocartesian S^{op}-family of symmetric monoidal ∞-categories, and let $q : \mathcal{D}^{\otimes} \to \mathcal{S}\mathrm{urj}_*$ be a presentably symmetric monoidal ∞-category.

Definition B.1. Applying Definition 6.8 to the cocartesian fibrations p_1 and $S^{op} \times q$, we get a map of simplicial sets

$$\mathrm{Fun}(p_1,q) : \mathrm{Fun}_{S^{op}\times \mathcal{S}\mathrm{urj}_*}(\mathcal{C}^{\otimes}, S^{op}\times \mathcal{D}^{\otimes}) \to S^{op}\times \mathcal{S}\mathrm{urj}_*.$$

By [11, 2.4], which, as Glasman notes, generalizes immediately to cocartesian fibrations over an arbitrary base, the map $\mathrm{Fun}(p_1,q)$ is a locally cocartesian fibration. Moreover, a morphism in

$$\mathrm{Fun}_{S^{op}\times \mathcal{S}\mathrm{urj}_*}(\mathcal{C}^{\otimes}, S^{op}\times \mathcal{D}^{\otimes})$$

is locally $\mathrm{Fun}(p_1,q)$-cocartesian if and only if represents a certain relative left Kan extension.

The desired cocartesian fibration $p_2 : \mathrm{Fun}_{S^{op}}(\mathcal{C},\mathcal{D})^{\otimes} \to S^{op} \times \mathcal{S}\mathrm{urj}_*$ will be the restriction of $\mathrm{Fun}(p_1, q)$ to a certain subcategory of

$$\mathrm{Fun}_{S^{op}\times \mathcal{S}\mathrm{urj}_*}(\mathcal{C}^{\otimes}, S^{op} \times \mathcal{D}^{\otimes}).$$

This subcategory consists of those simplices that 'decompose' relative to the wedge sum operation in the category $\mathcal{S}\mathrm{urj}_*$.

Lemma B.2. *For a finite pointed set J_+, there is a pullback square*

$$\begin{array}{ccc} \prod_{S^{op}, j\in J_+} \mathcal{C}^{\otimes} & \longrightarrow & \mathcal{C}^{\otimes} \\ \downarrow{\scriptstyle \prod_j p_1} & & \downarrow{\scriptstyle p_1} \\ \prod_{S^{op}, j\in J_+} S^{op} \times \mathcal{S}\mathrm{urj}_* & \xrightarrow{\vee_{j\in J_+}} & S^{op} \times \mathcal{S}\mathrm{urj}_* \end{array}$$

where the bottom horizontal map is given by the wedge sum of finite pointed sets, and the products on the left-hand side are in the category of simplicial sets over S^{op}.

Proof. The pullback over a product is a product of the pullbacks over each individual factor, where each factor maps into the product via the terminal object of $\mathcal{S}\mathrm{urj}_*$. But the restriction of the wedge sum to a single nontrivial factor is the identity, so the given pullback is simply a product of copies of $\mathcal{C}^{\otimes}$. □

Definition B.3. For an n-simplex $\sigma : \Delta^n \to \mathcal{S}\mathrm{urj}_*$ given by a sequence of surjections of finite pointed sets

$$(I_1)_+ \to (I_2)_+ \to \cdots \to (I_n)_+ \to J_+$$

we write $\sigma_j : \Delta^n \to \mathcal{S}\mathrm{urj}_*$ for the n-simplex given by the iterated inverse images of $j \in J_+$. (Thus if j is not the basepoint in J, then σ_j consists of a sequence of active morphisms mapping finally into $\{j\}_+$. If j is the basepoint, then σ_j is a sequence of arbitrary morphisms mapping finally into $*$.)

Lemma B.4. *Given an n-simplex $(f,\sigma) : \Delta^n \to S^{op} \times \mathcal{S}\mathrm{urj}_*$, let us write*

$$\mathcal{C}^{\otimes}_{f,\sigma} := \Delta^n \times_{S^{op}\times\mathcal{S}\mathrm{urj}_*} \mathcal{C}^{\otimes}.$$

Then there is an isomorphism

$$\mathcal{C}^{\otimes}_{f,\sigma} \cong \prod_{j\in J_+} \mathcal{C}^{\otimes}_{f,\sigma_j}$$

Proof. Consider the diagram

$$\begin{array}{ccccc}
\mathcal{C}^{\otimes}_{f,\sigma} & \longrightarrow & \prod_{S^{op},j\in J_+} \mathcal{C}^{\otimes} & \longrightarrow & \mathcal{C}^{\otimes} \\
\downarrow & & \downarrow {\scriptstyle \prod_j p_1} & & \downarrow {\scriptstyle p_1} \\
\Delta^n & \xrightarrow{(f,\sigma_j)_{j\in J_+}} & \prod_{S^{op},j\in J_+} S^{op}\times \mathcal{S}\mathrm{urj}_* & \xrightarrow{\vee_{j\in J_+}} & S^{op}\times \mathcal{S}\mathrm{urj}_*
\end{array}$$

where the right-hand pullback square is as in Lemma B.2. Since the overall square is a pullback too, so is the left-hand square, which induces the desired isomorphism. □

Remark B.5. When $S = *$, the argument of Lemma B.4 reduces to that of [11, 2.6]. In particular, there is a corresponding isomorphism

$$\mathcal{D}^{\otimes}_{\sigma} \cong \prod_{j\in J_+} \mathcal{D}^{\otimes}_{\sigma_j}.$$

Definition B.6. Now consider an n-simplex in $\mathrm{Fun}_{S^{op}\times\mathcal{S}\mathrm{urj}_*}(\mathcal{C}^{\otimes}, S^{op}\times\mathcal{D}^{\otimes})$, consisting of an n-simplex

$$(f,\sigma):\Delta^n \to S^{op}\times\mathcal{S}\mathrm{urj}_*$$

and a map

$$G:\mathcal{C}^{\otimes}_{f,\sigma}\to\mathcal{D}^{\otimes}_{\sigma}$$

over Δ^n. Let J_+ be the final vertex of σ as above. We then say that $((f,\sigma),G)$ is *decomposable* if it is the product of n-simplices of the form $((f,\sigma_j),G_j)$ for $j\in J_+$, with respect to the isomorphisms in B.4 and B.5, i.e. if it factors as

$$\mathcal{C}^{\otimes}_{f,\sigma}\cong\prod_{S^{op},j\in J_+}\mathcal{C}^{\otimes}_{f,\sigma_j}\subseteq\prod_{j\in J_+}\mathcal{C}^{\otimes}_{f,\sigma_j}\xrightarrow{\prod_{j\in J_+}G_j}\prod_{j\in J_+}\mathcal{D}^{\otimes}_{\sigma_j}\cong\mathcal{D}^{\otimes}_{\sigma}.$$

Example B.7. A vertex in $\mathrm{Fun}_{S^{op}\times\mathcal{S}\mathrm{urj}_*}(\mathcal{C}^{\otimes}, S^{op}\times\mathcal{D}^{\otimes})$, i.e. a map $G:\mathcal{C}^{\otimes}_{s,\langle n\rangle}\to\mathcal{D}^{\otimes}_{\langle n\rangle}$, is decomposable if and only if it is in the image of the map

$$\mathrm{Fun}(\mathcal{C}_s,\mathcal{D})^n\to\mathrm{Fun}(\mathcal{C}_s^n,\mathcal{D}^n)\cong\mathrm{Fun}(\mathcal{C}^{\otimes}_{s,\langle n\rangle},\mathcal{D}^{\otimes}_{\langle n\rangle}).$$

Definition B.8. Let $\mathrm{Fun}_{S^{op}}(\mathcal{C},\mathcal{D})^{\otimes}$ be the simplicial subset of $\mathrm{Fun}_{S^{op}\times\mathcal{S}\mathrm{urj}_*}(\mathcal{C}^{\otimes}, S^{op}\times\mathcal{D}^{\otimes})$ consisting of those simplices for which all the faces are decomposable, and let

$$p_2:\mathrm{Fun}_{S^{op}}(\mathcal{C},\mathcal{D})^{\otimes}\to S^{op}\times\mathcal{S}\mathrm{urj}_*$$

be the restriction of $\mathrm{Fun}(p_1,q)$ to this subset.

When $S = *$, Definition B.8 reduces to Glasman's construction. It follows that the fibres of the map p_2 are precisely the symmetric monoidal ∞-categories $\mathrm{Fun}(\mathcal{C}_s,\mathcal{D})^{\otimes}$. The next result verifies most of the remaining requirements of Proposition 6.17.

Proposition B.9. *The map p_2 of Definition B.8 is a cocartesian family of symmetric monoidal ∞-categories.*

Proof. Glasman's proofs of Lemmas 2.9, 2.10 and 2.11 of [11] extend to the case $S \neq *$ and imply that p_2 is a cocartesian fibration, and an S^{op}-family of ∞-operads. □

It remains to show that the family p_2 is stable, and that the multi-morphism spectra are given by the formulas in the statement of Proposition 6.17. Since we already know that the fibres of p_2 are stable symmetric monoidal ∞-categories, both these claims follow from the next calculation.

Lemma B.10. *The multi-morphism spaces for the family of ∞-operads p_2 take the form*

$$\mathrm{Hom}_{\mathrm{Fun}_{S^{op}}(\mathcal{C},\mathcal{D})^{\otimes}}(A_1,\dots,A_n;B)_f \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}_{s'},\mathcal{D})}(\mathcal{C}^!_f(A_1\otimes\cdots\otimes A_n);B)$$

for an edge $f : s \to s'$ in S.

Proof. This claim follows from the characterization of the locally cocartesian edges in p_2 in terms of left Kan extensions. □

Appendix C. Fibrewise duals for cocartesian fibrations

The goal of this section is to prove Proposition 6.19. Recall that the opposite of a symmetric monoidal ∞-category is built from the following general construction of [4].

Definition C.1. Let $q : Y \to T$ be a cocartesian fibration. Then there is a cartesian fibration

$$q^\vee : Y^\vee \to T^{op}$$

that classifies the same diagram of ∞-categories as q.

The construction $q \mapsto q^\vee$ is functorial in the following way. Let $p : Y \to Y'$ be a map between cocartesian fibrations over T that takes cocartesian edges to cocartesian edges. Then p induces a map

$$p^\vee : Y^\vee \to Y'^\vee$$

between cartesian fibrations over T^{op} that takes cartesian edges to cartesian edges.

Definition C.2. Let $p : Y \to S^{op} \times T$ be a map of simplicial sets such that:

- for each $s \in S$, the fibre $p_s : Y_s \to T$ is a cocartesian fibration;
- each p_s-cocartesian lift of a morphism in T is also p-cocartesian.

It follows that p is a map between cocartesian fibrations over T that takes cocartesian edges to cocartesian edges. Therefore p induces a map

$$p^\vee : Y^\vee \to (S^{op} \times T)^\vee \cong S^{op} \times T^{op}$$

between cartesian fibrations over T^{op} that takes cartesian edges to cartesian edges.

Taking opposites, we obtain another map

$$p^{\vee,op} : Y^{\vee,op} \to S \times T$$

between cocartesian fibrations over T that takes cocartesian edges to cocartesian edges.

Definition C.3. Now let $p : \mathcal{M}^{\otimes} \to S^{op} \times \mathcal{S}\mathrm{urj}_*$ be a stable cocartesian S^{op}-family of symmetric monoidal ∞-categories. Then p satisfies the conditions in Definition C.2, and so determines a map

$$p^{\vee,op} : \mathcal{M}^{op,\otimes} \to S \times \mathcal{S}\mathrm{urj}_*$$

between cocartesian fibrations over $\mathcal{S}\mathrm{urj}_*$ that takes cocartesian edges to cocartesian edges.

We now check that $p^{\vee,op}$ satisfies the requirements of Proposition 6.19.

Proposition C.4. *The map $p^{\vee,op} : \mathcal{M}^{op,\otimes} \to S \times \mathcal{S}\mathrm{urj}_*$ constructed in Definition C.3 is a stable S-family of symmetric monoidal ∞-categories. The fibre of $p^{\vee,op}$ over $s \in S$ is the opposite symmetric monoidal ∞-category $(p_s)^{\vee,op} : \mathcal{M}_s^{op,\otimes} \to \mathcal{S}\mathrm{urj}_*$ and the multi-morphism spectra are given, for $f : s \to s'$ in S, by*

$$\mathrm{Map}_{\mathcal{M}^{op,\otimes}}(X_1, \dots, X_n; Y)_f \simeq \mathrm{Map}_{\mathcal{M}_s}(\mathcal{M}_f(Y), X_1 \otimes \dots \otimes X_n)_{f^{op}}.$$

Proof. We check conditions (1)-(3) of 6.1. For (1), the construction implies that the fibre over $s \in S$ is, as claimed, the opposite symmetric monoidal ∞-category $\mathcal{M}_s^{op,\otimes}$.

Since $\mathcal{M}^{op,\otimes} \to \mathcal{S}\mathrm{urj}_*$ is a cocartesian fibration, each morphism $\alpha : \langle m \rangle \to \langle n \rangle$ has a cocartesian lift $\bar{\alpha}$ which maps to an edge of $S \times \mathcal{S}\mathrm{urj}_*$ that is also cocartesian over $\mathcal{S}\mathrm{urj}_*$. It follows from (the dual of) [17, 2.4.1.3(3)] that $\bar{\alpha}$ is also $p^{\vee,op}$-cocartesian, which implies (2).

For (3), consider objects $(X_1, \dots, X_n) \in \mathcal{M}_s$ and $(Y_1, \dots, Y_m) \in \mathcal{M}_{s'}$, a 1-simplex $f : s \to s'$ in S, and a morphism $\alpha : \langle n \rangle \to \langle m \rangle$ in $\mathcal{S}\mathrm{urj}_*$.

We then have

$$\mathrm{Hom}_{\mathcal{M}^{op,\otimes}}((X_1, \dots, X_n), (Y_1, \dots, Y_m))_f \simeq \mathrm{Hom}_{(\mathcal{M}^{\otimes})^{\vee}}((Y_1, \dots, Y_m), (X_1, \dots, X_n))_{f^{op}}.$$

Since the cartesian fibration $(\mathcal{M}^{\otimes})^{\vee} \to \mathcal{S}\mathrm{urj}_*^{op}$ encodes the same diagram of ∞-categories and functors as the cocartesian fibration $\mathcal{M}^{\otimes} \to \mathcal{S}\mathrm{urj}_*$, the part of the above mapping space over α is equivalent to

$$\mathrm{Hom}_{\mathcal{M}^{\otimes}_{\langle m \rangle}}((Y_1, \dots, Y_m), \mathcal{M}^{\otimes}_{\alpha}(X_1, \dots, X_n))_{f^{op}}$$

which is equivalent to

$$\prod_{i=1}^{m} \mathrm{Hom}_{\mathcal{M}^{\otimes}_{\langle 1 \rangle}}(Y_i, \bigotimes_{\alpha(j)=i} X_j)_{f^{op}}$$

and hence to

$$\prod_{i=1}^{m} \mathrm{Hom}_{\mathcal{M}_s}(\mathcal{M}_f(Y_i), \bigotimes_{\alpha(j)=i} X_j).$$

Using this decomposition, we can verify part (3) of 6.1, complete the proof that the family $\mathcal{M}^{op,\otimes}$ is stable, and obtain the desired description of its multi-morphism spectra. ∎

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